On the Forms of Plane Quartic Curves.

By [illegible] Bancroft.

Table of contents.

It is not my intention in this paper to consider the forms of quintic curves in general, or to give anything like a complete classification of such. I have confined myself principally to the determination of non-singular quintics [illegible] fifteen real inflections. These may be considered as fundamental, inasmuch as they serve to illustrate the forms of quintics having a less number of inflections. The method used being applicable to curves of all orders, I make a general statement of it; in so doing I [illegible] [illegible] his memoir on "Higher Plane [illegible]." I must add also that in the practical application of the method I have done little or nothing more than to f[illegible] out Dr Story's suggestions.

If $S=0$ represent a curve of any order, [illegible]

[illegible] case of a double [illegible] curve of lower order, we pass from a proper curve to an improper one. Conversely, given an improper curve, we can by a change of parameters pass from it to a proper curve of the same order which may be considered its neighbor. The new curve may or may not have points in common with the old and will be near or far from it, according as the change of parameters has been small or large. A change involving the absolute term only gives a curve having no points in common with the curve from which it is derived, and furnishes at once the simplest and most important method for determining the singular forms of a given curve. Such a change is sufficient for many purposes; changes in others will be considered.

In the study of a curve we assume the operations of [illegible].

particularly true of non-singular curves. But, if we have given a series of curves of lower orders, we can form improper curves of higher orders, and, from these, it is a comparatively simple matter to determine non-singular curves of a higher order by the general construction which follows. I shall confine my attention to the case of the construction of two proper nonsingular curves. Representing these curves by u and v and the improper curve resulting from their composition by uv, the desired non-singular curve will be represented by $uv = k$. It will always be supposed that k is small enough, not necessarily infinitesimal, to avoid gaining or passing through any singularities in going from uv to $uv = k$. The intersections of u and v will be referred to as double points of uv.

Inflections of $uv = k$.

We know that in gaining a [illegible]

[illegible]
the loss of a double point involves the gain of two real inflexions. In passing from [illegible] to $uv = k$, all the double points of uv are lost, [illegible]

Dr. Story has remarked that the [illegible] curve is probably one third of the total [illegible]

[illegible]

n_2 the order of a curve v, and suppose they intersect in the maximum number of points, $n_1 n_2$. The number of inflections on curve uv will be made up of the number on u, the number on v, and the number arising from the loss of double points of uv; i.e. $n_1(n_1-2) + n_2(n_2-2) + 2n_1 n_2 = n_1^2 + 2n_1 n_2 + n_2^2 - 2(n_1+n_2)$ $= (n_1+n_2)(n_1+n_2-2)$ which is the number required. Now we know that what has been stated is true of curves of orders 2, 3, and 4, hence, by by what has just been proved it is true generally. It will be observed that as the points of intersection of u and v diminish in sets of two, the inflections of the corresponding non singular curves will be diminished in sets of four; so that our curves will have, if we form a complete system by this method, $n(n-2)$, $n(n-2)-4$, $n(n-2)-8$ &c to 3 or 0 inflections according as n is odd or even.

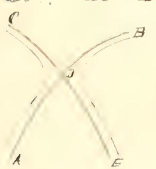
Bitangents of uv [illegible] examined as derived from the bitangents of u, v.

As a line passing through a double point satisfies the analytical conditions for tangency, it may be regarded as a tangent, and to distinguish it from ordinary tangents, we call it an analytical tangent. So, in the case of bitangents to a singular curve we have, in addition to the ordinary bitangents, analytical bitangents. The bitangents may be considered as falling under three classes; (a) tangents having two real or imaginary points of contact = the ordinary bitangents; (b) lines through double points tangent elsewhere; (c), lines joining double points two and two. In the case of an improper singular curve uv, class (b) is divided into two; (b_1) lines through the double points tangent to u; (b_2) lines through the double points tangent to v; the number in each of these is two less than the class of the curve to which

multiplied by the number of double points,
the tangents are drawn, since a tangent at a double point . . . counts for two and is not regarded as a bitangent.

Since $uv=k$ is derived from uv, the bitangents of $uv=k$ may be regarded as derived from those of uv. Each ordinary bitangent of uv is [illegible].

To determine the number due to the analytical bitangents we must investigate the nature of $uv=k$ in the neighborhood of the double points of uv. Let D be a double point of uv. For any



particular value of k the product [illegible] sign, namely, the sign of k, [illegible] must change sign [illegible] at all. A simultaneous change of sign can take place only when we pass from an angle ADC to its opposite angle BDE. It follows, therefore, that if a part of the curve $uv=k$ is in the angle [illegible]

there will be a corresponding part in the angle DBE. If the sign of k be changed then parts of the [illegible] are in the opposite angles ABE and CDB.

To investigate the nature of $uv=k$ as to the direction of its points relative to uv, let x and y be coordinates of any point on uv and ε_1 and ε_2 the increments of x and y which will bring us from a point on uv to a point on $uv=k$. Substituting [illegible] the equation $uv=k$ will be satisfied. The [illegible] theorem [illegible]

$$\left(u+\varepsilon_1\frac{\partial u}{\partial x}+\varepsilon_2\frac{\partial u}{\partial y}+\text{etc}\right)\left(v+\varepsilon_1\frac{\partial v}{\partial x}+\varepsilon_2\frac{\partial v}{\partial y}+\text{etc}\right)=k$$

If the point x, y is on only one of the curves u, v say [illegible] then [illegible], leaving out higher powers of [illegible] and [illegible] [illegible]

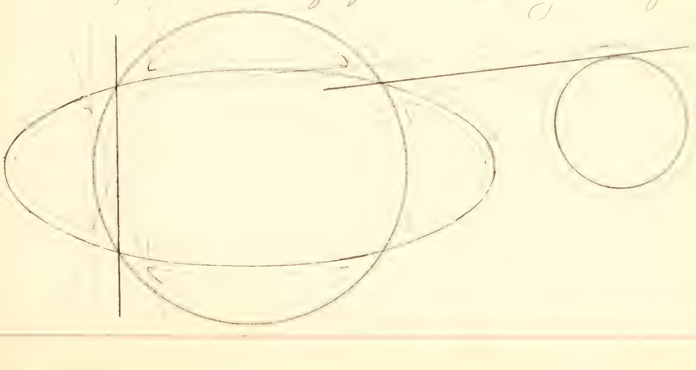
[illegible]

the equation of a line parallel to the tangent [illegible] at the point [illegible]. This line gives the direction of an element of the curve $uv=k$. [illegible]

a double point on uv then $u=0$ and $v=0$. Omitting powers of ε_1 and ε_2 higher than the second, we have

$$\left(\varepsilon_1 \frac{\partial u}{\partial x} + \varepsilon_2 \frac{\partial u}{\partial y}\right)\left(\varepsilon_1 \frac{\partial v}{\partial x} + \varepsilon_2 \frac{\partial v}{\partial y}\right) = k ,$$

which shows that in the neighborhood of a double point of [illegible] approximately [illegible] asymptotes [illegible] at the point in question. [illegible] the position of the [illegible] and [illegible] that a double point gives rise to two real inflections. With what has been proved a figure will serve to show that each analytical bitangent of the class (b) is the source of two, and each of class (c), the source of four [illegible] bitangents.



It will be observed that in some cases the [illegible] bitangents of classes (b) and (c) must be regarded as imaginary; such a case is indicated by the dotted curves in the figures.

We can now determine the total number of bitangents to $uv = k$.

Let m_1 denote the class of uv,

m_2 " " " u

m_3 " " " v

n " order " uv

δ " number of double points of uv, taken as the maximum.

Let τ denote the number of ordinary bitangent of [illegible]

τ_1 " " " " " " uv

τ_2 " " " bitangents of $uv = k$ arising from tangents drawn to u from intersections of u and v

τ_3, the number of bitangents [illegible] arising from tangents drawn to v from intersections of u and v

τ_4, the number of bitangents arising from those joining the intersections of u and v two and two.

When $\varkappa$ the number of cusps is zero, [illegible] the class $m = n(n-1) - 2\delta$ and the number of bitangents [illegible] $\frac{1}{2}(m^2 - 10m + 8n)$.

[illegible] Suppose first that n is even. Taking [illegible] of the same order $\frac{n}{2}$ [illegible]

$[illegible] \quad m_1 = m_2 = \frac{n(n-2)}{4}$.

$\tau_1 = \frac{1}{2}\left[\frac{n^2(n-1)^2}{4} - 5n(n-1) + 8n\right] = \frac{n}{2}\left([illegible] + 18 \right)$

$\tau_2 = \tau_3 = (m_1 - 2)\frac{n^2}{4} = \frac{n^2}{2}\left[\frac{n}{2}\left(\frac{n-2}{2}\right) - 2\right] = [illegible]$

$\tau_4 = 4\frac{n}{2}\left([illegible] - 1\right) = [illegible]$

$[illegible] = \frac{n}{2}\left(n^3 - 2n^2 - 9n + 18\right)$

$= \frac{n}{2}(n-2)(n^2-9)$

which is the [illegible] of bitangents of [illegible] of the n^{th} order.

If n is odd [illegible] of order $\frac{n-1}{2}$ and [illegible] of order $\frac{n+1}{2}$ [illegible]

$[illegible] = \frac{n^2-1}{4}, \quad m = n(n-1) - \frac{n^2-1}{2} = \frac{(n-1)^2}{2} \quad [illegible] = \frac{n^2-1}{4}$.

$\tau_1 = [illegible] = \frac{1}{2}\left([illegible] - \frac{19}{4}\right)$

$\tau_2 = [illegible] = \frac{1}{2}\left([illegible] + \frac{5}{4}\right)$

$\tau_3 = \left(\frac{n^2-1}{4} - 2\right)\frac{n^2-1}{2} = \frac{1}{2}\left(\frac{n^4}{4} [illegible] + \frac{1}{4}\right)$

* The investigation can be made general with the same result.

$$\tau_4 = \frac{n^2-1}{4}\left(\frac{\frac{n^2-1}{4}-1}{1.\ 2}\right)4 = \frac{1}{2}\left(\frac{n^4}{4}-\frac{3}{2}n^2+\frac{5}{4}\right)$$

$$\tau = \tau_1+\tau_2+\tau_3+\tau_4 = \frac{1}{2}(n^4-2n^3-9n^2+18n)$$

$$= \frac{n}{2}(n-2)(n^2-9)$$

which, as before, is the total number of bitangents to a curve of the n^{th} order.

Some properties of non-singular curves which [illegible] classification of non-singular curves in general.

Zeuthen has remarked (Math. Annalen Vol 7 p 410) that the branches that make up a plane curve may be of two kinds, odd or even. The peculiarity of an odd branch is, that it intersects every other odd branch in an odd number of points; the even branch intersects any other branch in an even number of points. Zeuthen has shown that an odd branch has always an odd number of real inflections and an even branch an even number, and, if an odd branch is non-singular it must have at least three real inflexions. [illegible]

Grundformen der Linien Dritter Ordnung §§ 10, 11 and 12.

An even branch will be called after Zeuthen an odd, an unipartite, a bipartite, a tripartite or [illegible] according as it has [illegible], [illegible] or [illegible] real inflexions.

A non-singular quartic cannot have more than one odd branch for, if it had more than one, it would have at least one double point. An odd branch must have at least three real inflexions. Quartics [illegible] of the [illegible] already [illegible] [illegible] these [illegible] [illegible] [illegible] real inflections. They would [illegible] a combination of a cubic and a conic. A quartic cannot have more than six even branches external to each other, for, if there were more, say seven, through a point in each of these [illegible] [illegible] [illegible] [illegible]. All branches [illegible] could have [illegible] [illegible] intersecting the quartic in at least [illegible] point, since the cubic must intersect each even [illegible] in

at least one other point, besides the point chosen. There cannot be more than two even branches [illegible] of the other; [illegible] branch must be an oval; otherwise we could draw a straight line intersecting the quintic in at least seven points.

Zeuthen divides bitangents into two kinds; his bitangents of the first kind are those whose contacts are on the same branch, the contacts [illegible]; bitangents of the second kind are those whose contacts are on different branches. In the case of quartics the contacts of a bitangent of the first kind cannot be [illegible] by its intersection with the other bitangents. This is true of quintics if the bitangent of the first kind has its contacts on an even branch. The contrary may be the case if the contacts are on an odd branch.

[illegible]

tangent elsewhere. It is easily seen that the number of bitangents of the first kind to an even branch is one half of the number of its real inflections. Further there may be 2, 3, 4, 5 or 6 even branches; hence they may have 0, 0, 4, 12, 24, 40, 60 bitangents of the second kind to even branches, since every combination of even branches in sets of two have four bitangents of the second kind. The other bitangents of the second kind are those whose contacts are one on the old branch and one on an even branch. The total number of bitangents = one [illegible].

Enumeration of forms.

Since all non-singular quartics have one odd branch it is sufficient to state for each case the characters of the even branches. I have found it convenient ~~to~~ to group the curves, first, according to the total number of real inflections and then [illegible]

To the number of inflections in the same branch. I gave figures only for branches having fifteen real inflections. The following are the names of the same branches

Sixteen inflections in the same branch.

1. Bifolium and interior oval. Figs. 1 & 7.
2. Two unifolia. Figs. 1, 2, 3.

Ten inflections on the same branch.

3. A quadrifolium and a unifolium. Fig. 5.
4. A quadrifolium, a unifolium and an oval. Fig. 6.
5. A trifolium and a bifolium. Fig. 7.
6. A trifolium, a bifolium and an oval. Fig. 8.
7. [illegible]-folium and two ovals. Fig. 8.
8. A bifolium and two unifolia. Figs. 9, 10, 11, 12, 13.
9. A bifolium, three unifolia and an oval. Figs. 14, 15.
10. Two bifolia, a unifolium and two ovals. Fig 16
11. Five unifolia Figs. 17, 18, 19.

Eight inflections in the same branches

12. A quadrifolium and an oval Figs. 20, 21.

13. A quadrifolium and two ovals Fig. 22.

14 A trifolium and a unifolium. Figs. 23, 24.

15. A trifolium, a unifolium and an oval. Figs. 25, 26.

16. Two bifolia. Fig 27

17. Two bifolia and an oval Fig 28

18. Two bifolia and two ovals Figs. 29, 30

19. A bifolium and two [illegible] Figs 31, 32, 33 34, 35.

20 Four unifolia. Figs. 36, 37, 38

21. A quadrifolium and three ovals.

22. A bifolium, two unifolia and an oval.

The figures for 21 and 22 are not given; 21 results from a combination of a bipartite redundant hyperbola with a hyperbola; 22 from [illegible] cubic with an ellipse. The form of 22 will be readily seen from Figs. 33, 34, 35.

Six inflections on the even branches

23. A trifolium and internal oval. Fig. not drawn. Figs. 17, 18 ... will illustrate this case.

24. A bifolium ...

25. A trifolium and an oval. Fig. [illegible]

26. A trifolium and two ovals. Figs 13, [illegible]

27. A trifolium and three ovals. Fig. [illegible]

28. A bifolium and a unifolium. Fig [illegible]

29. A bifolium a unifolium and an oval. Figs. 45, 46, 47.

30. Two unifolia Figs. 45, 46, 47.

31 Three unifolia and one oval. Fig [illegible]

Forms symmetrical as to [illegible]

32. A bifolium Figs. [illegible]

33. A bifolium and an oval. Figs. 10, 12, 42.

34. Two unifolia. Fig. 48

35. A bifolium and three ovals

36. Two unifolia and two ovals.

35 and 36 may be obtained by combining a bipartite redundant hyperbola with an ellipse.

Non-symmetrical — [illegible]

37. A unifolium and an oval. Figs [illegible]

38. A unifolium and three ovals. Formed as 35 and 36

by changing the position of the ellipse [illegible] the [illegible] so [illegible] so that the corresponding [illegible]

in the position as for cases 3[illegible], 3[illegible], 3[illegible].

Even branches without inflections.

39 A single oval. Figs. 5, 7, 21, 26.

40 Two ovals. Figs. 8, 11, 13, 27, 33, 34, 35, 36.

41 Three ovals. Figs. 37, 44, 51.

42 Four ovals [illegible] can be obtained as 38 by changing the positions of the ellipses.

No even branches.

43. Figs. 28, 48.

The figures for quintics with fifteen inflections will illustrate the forms that would result from combinations of a conic and cubic cutting in four, five or six points. The following are some of the most evident forms.

Quintics with eleven inflections.

Eight inflections on the even branches

1 A quadrifolium and an interior oval.

2 A quadrifolium and an oval.

3 Two trifolia and an oval.

4. Four unifolia

5. A trifolium and a unifolium.

6. A trifolium, a unifolium and an oval.

Five inflections on the even branches.

7. A trifolium.

8. A trifolium and an oval.

9. A trifolium and two ovals.

10. A trifolium and three ovals.

11. A bifolium and a unifolium.

12. A bifolium, a unifolium and an oval.

13. A bifolium, a unifolium and two ovals

14. Three unifolia.

15. Three unifolia and an oval.

Four inflections on the even branches.

16. A bifolium and an internal oval.

17. A bifolium and an oval.

18. A bifolium

19. Two unifolia.

Two inflections on one same branch.

20. A unifolium and an oval.

No inflections on even branches

21 An oval.

22 Two ovals

23 Three ovals

24 A single odd branch.

Quintics with seven inflections

Four inflections on even branches

1 A bifolium with external oval.

2 A bifolium and an oval.

3 A bifolium.

4 Two unifolia.

Two inflections on the same branch.

5 A unifolium with external oval.

6 A unifolium and an oval.

7 A unifolium.

No inflections on even branches.

9. Two ovals.

10 No even branches.

Quintics with three inflections

1 One oval and an internal oval.

2 Two ovals.

3 One oval.

The total number of forms enumerated is [illegible]

In order to show readily the characters of the even branches in each figure, I use the following notation. An oval is represented by o; a unifolium by 1; a bifolium by 2, &c. If the oval is inside of another even branch it is indicated by enclosing the o in parenthesis thus (o). A combination of digits will show at once the [illegible] of the even branches in question; these combinations [illegible] [illegible]

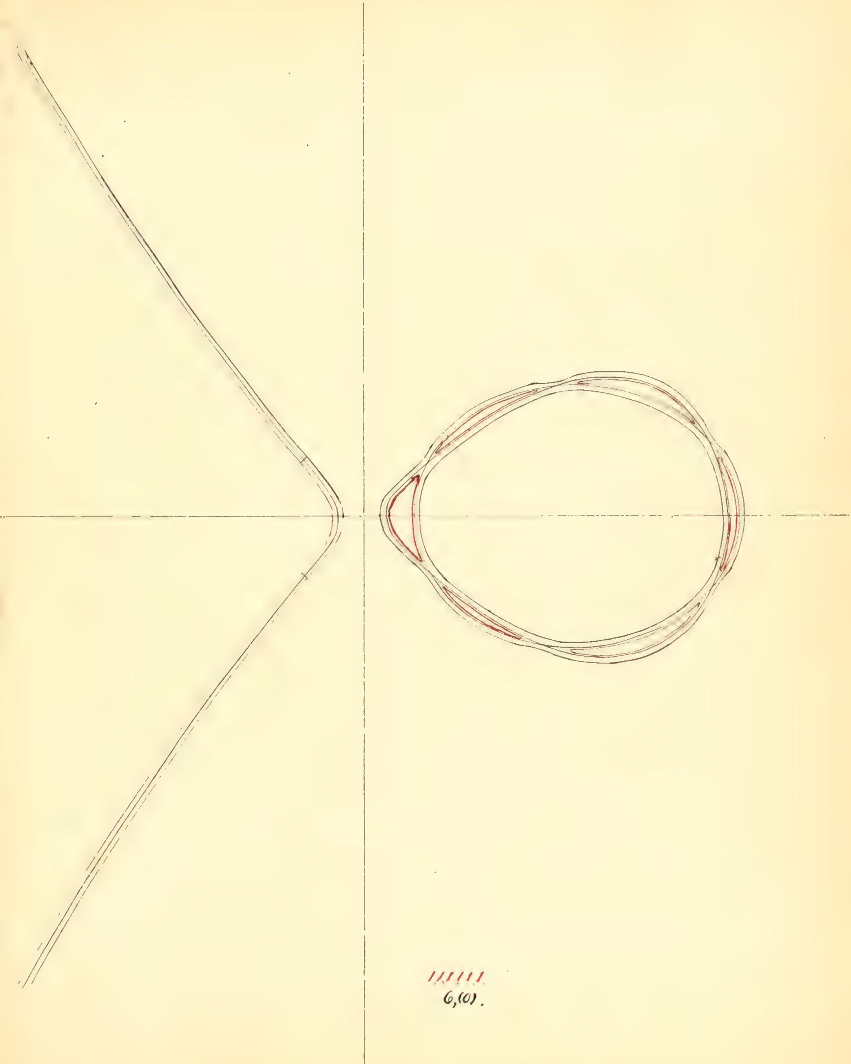
position is [illegible] in a blue [illegible]. The nature of the odd bounded [illegible] will appear on inspection. It will be seen that odd bounded [illegible] are composed of [illegible] three or five [illegible] parts having common [illegible] [illegible]. The dashes in the black [illegible] indicate the position of the [illegible] of the [illegible] these were determined [illegible] of the [illegible]. The position of the [illegible] of the [illegible] can be obtained [illegible] that, [illegible] of [illegible] of the [illegible] [illegible] angle having [illegible] one in an angle formed by a convex and a concave part and one in the opposite angle.

With regard to the real [illegible] I am una-

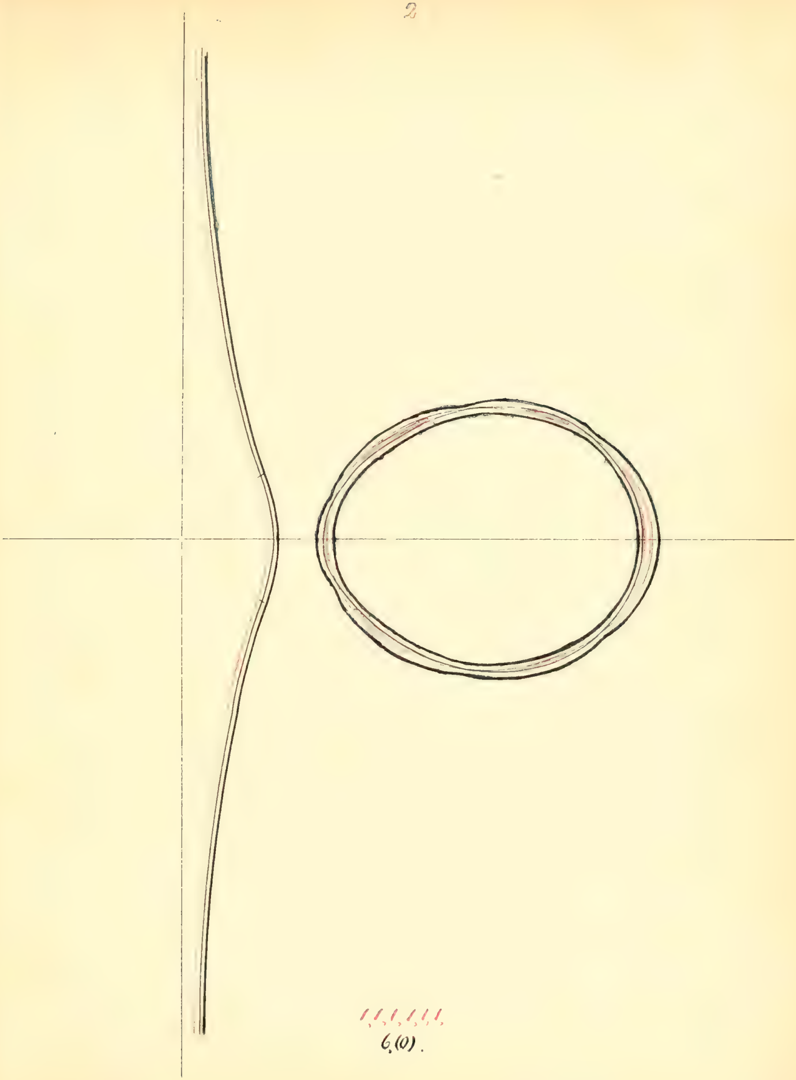
to state a general law as to their number. In the case of the quartic the number of bitangents of the first kind is constant but with the quintic ~~I believe~~ it [illegible] 0,0 [illegible] has twelve; 0, of Fig 7 has fifteen. A number of the curves [illegible] some have only eight and others no more than six seem to be evident. Of course [illegible] [illegible] of [illegible] [illegible] but [illegible] the case of a quintic with fifteen real inflections [illegible] Figs [illegible] [illegible] illustrate the loss of two bitangents of the first kind. 2,1,1,1 of Figs. 9 and 10 differ as to the position of the [illegible] of 10, is cut by the axis of [illegible] is an inflectional tangent [illegible] the cubic; with 9 it is not so. [illegible] [illegible] which [illegible] an inflectional tangent, tangent elsewhere. This inflectional bitan[illegible]

[illegible]

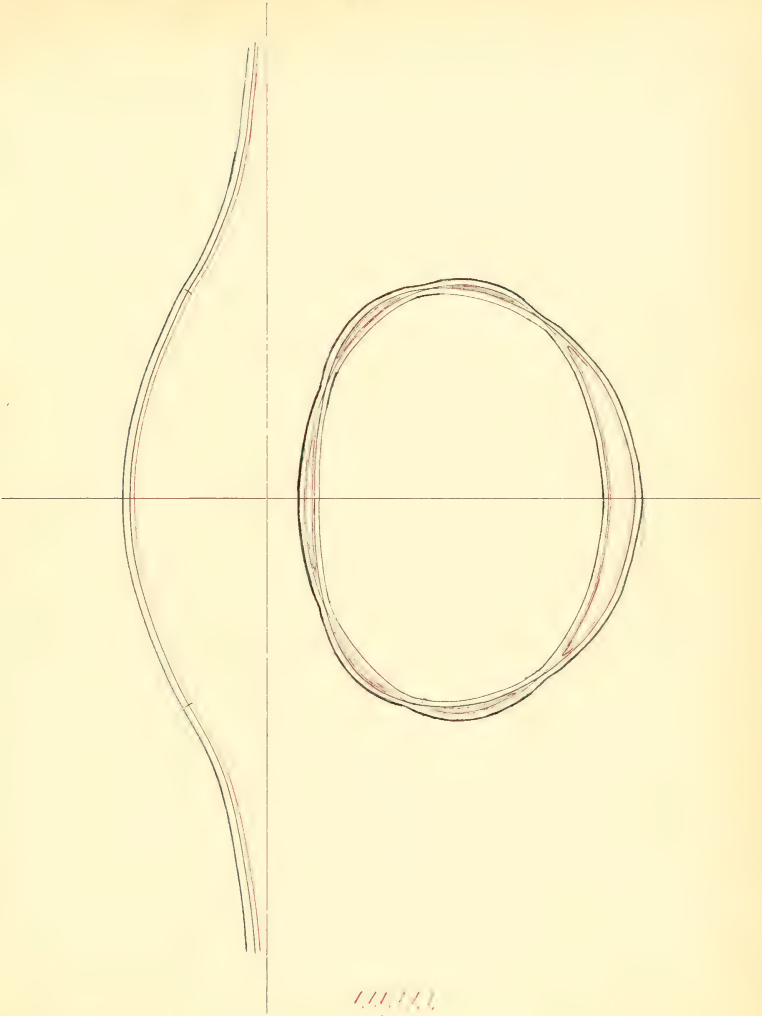
branch of Fig 9. Passing through the curve with an inflectional bitangent we come to the case of Fig. 10 which has two bitangents of the first kind less than Fig. 9. The same is true of and of Figs 15 and 17. The statement made on page 14, that the contacts of a bitangent of the first kind to an odd branch may be separated by the intersections of the bitangent with two others, is illustrated by 00, of Fig. 6 2, of Fig. 8, 3,0,0, of Fig. 43, 0,0, of Fig 44.



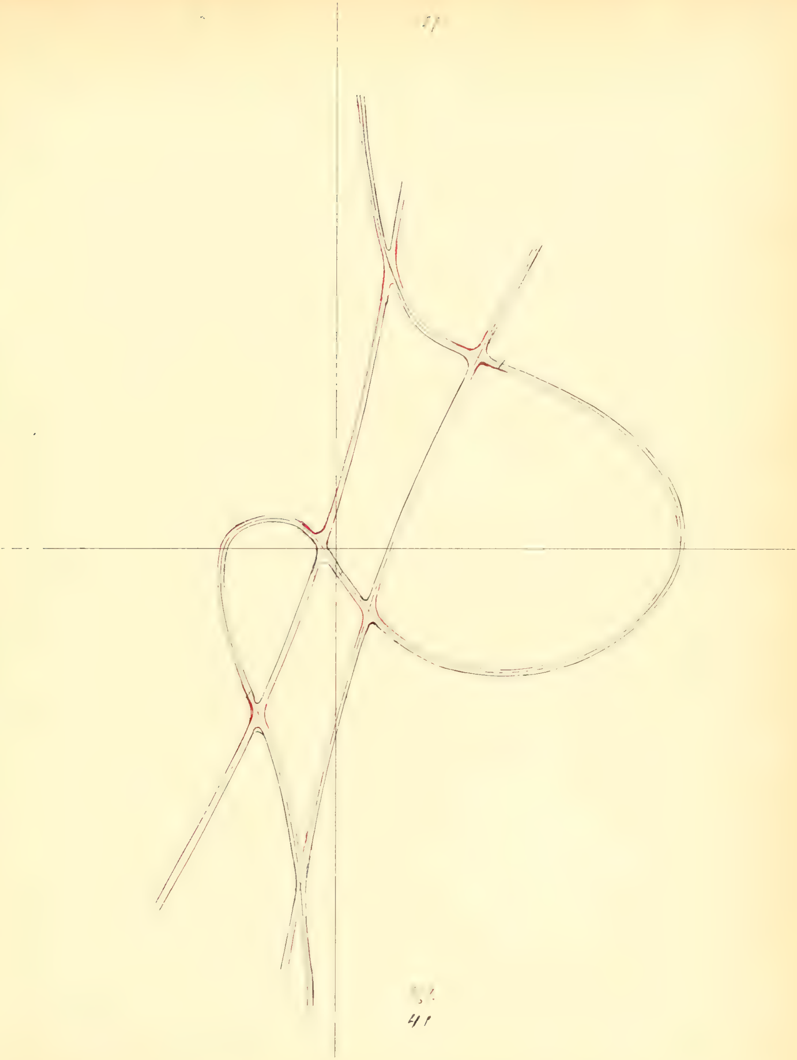
6,(0).



1, 1, 1, 1, 1, 1,
6, (0).



6,(0).



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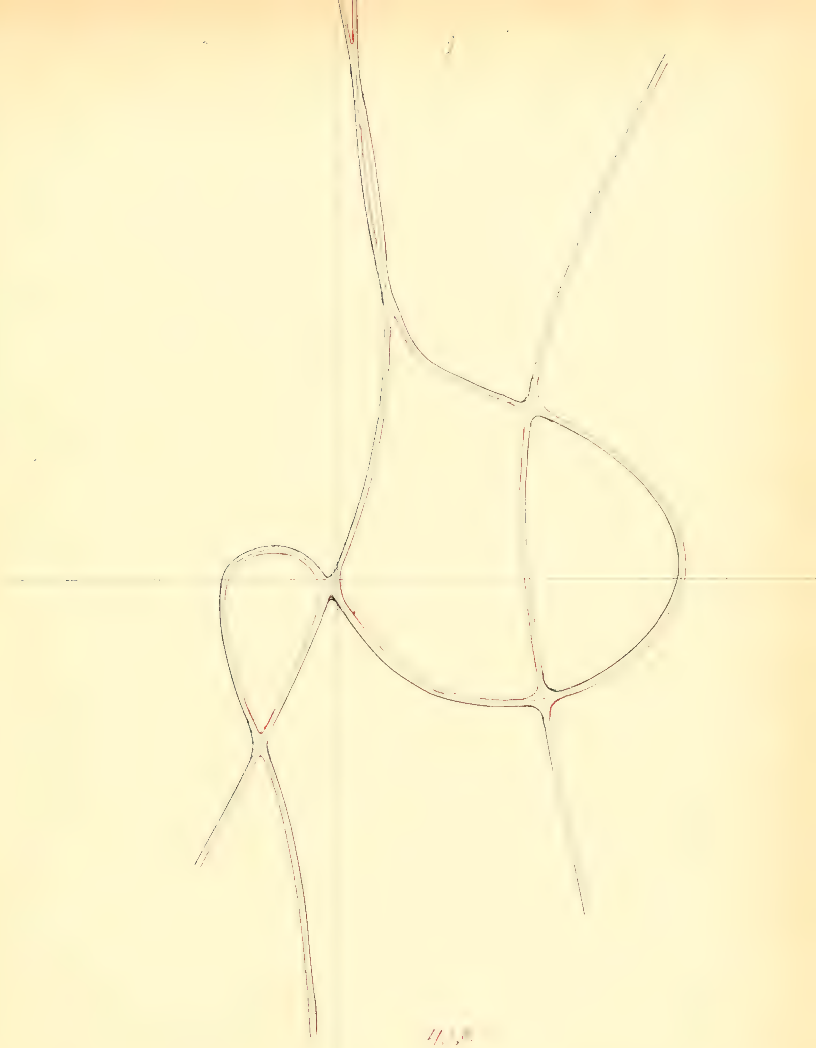

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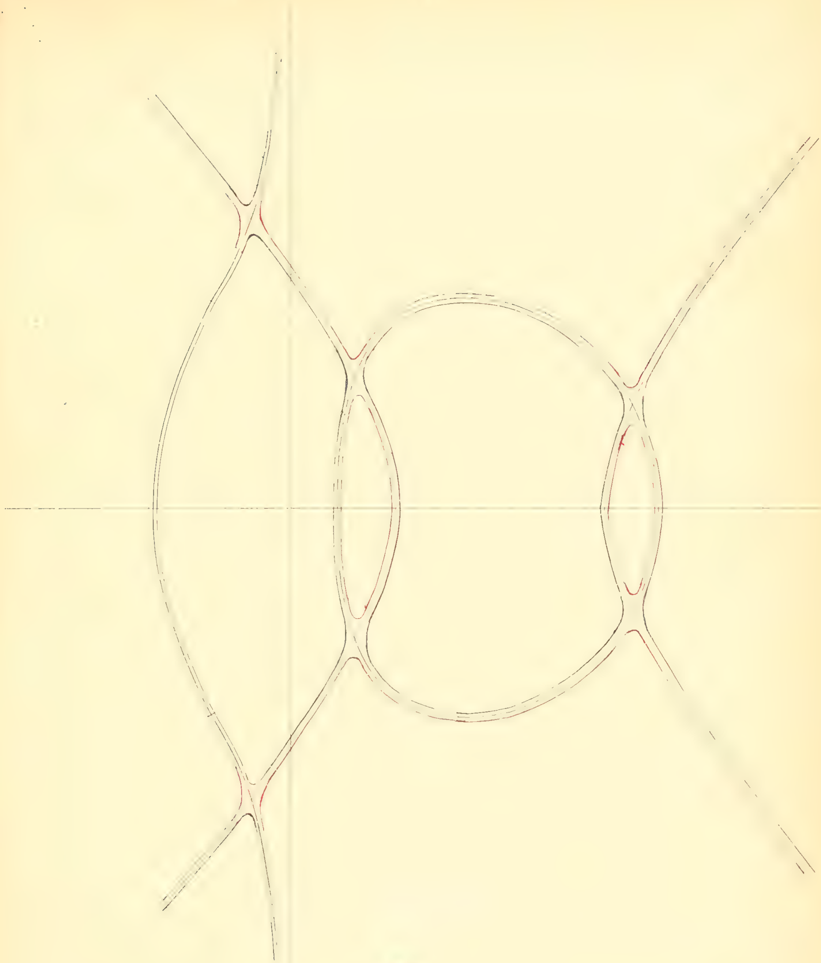

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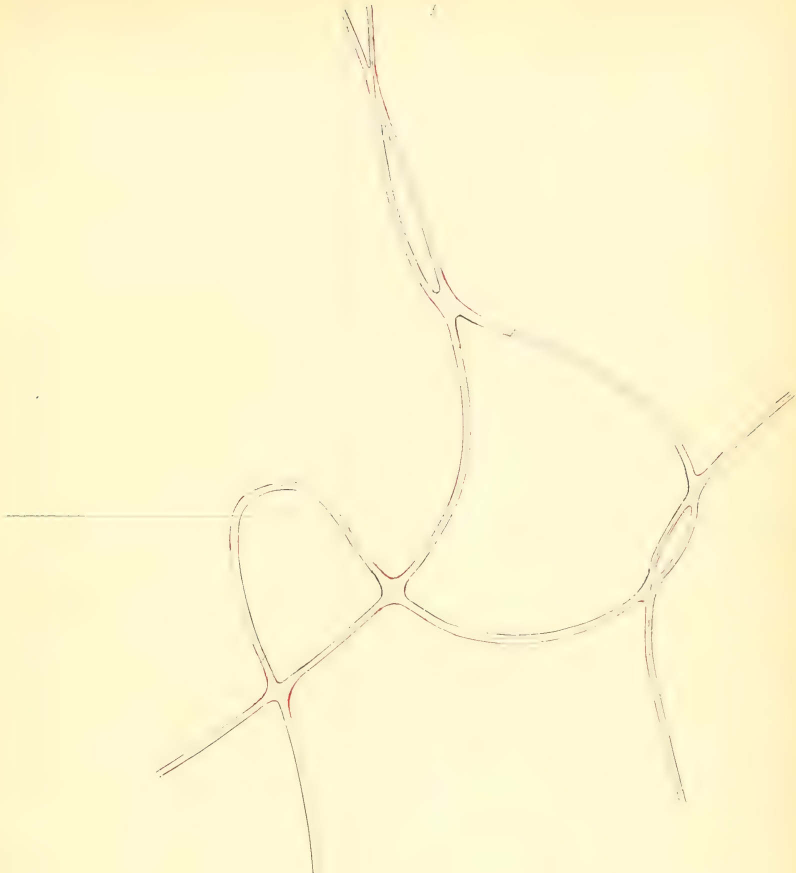
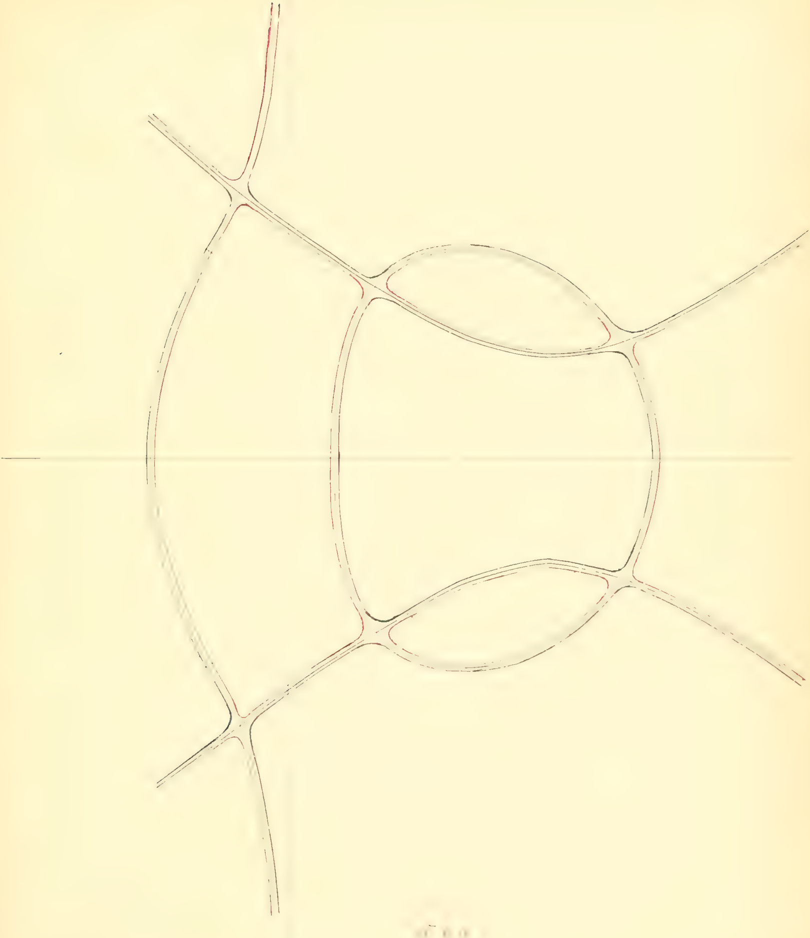
3,2.



0.
3,2,0.



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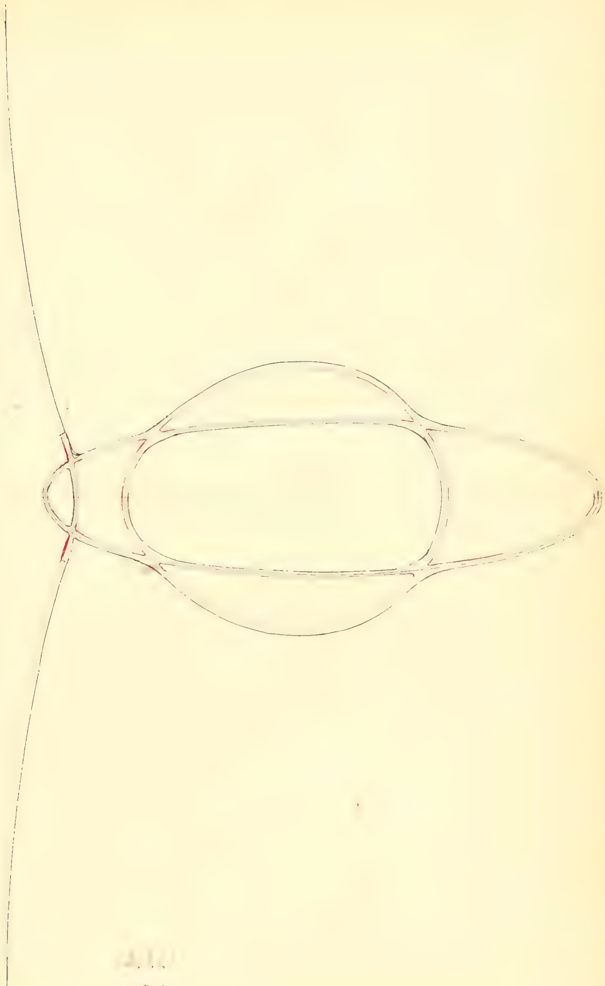
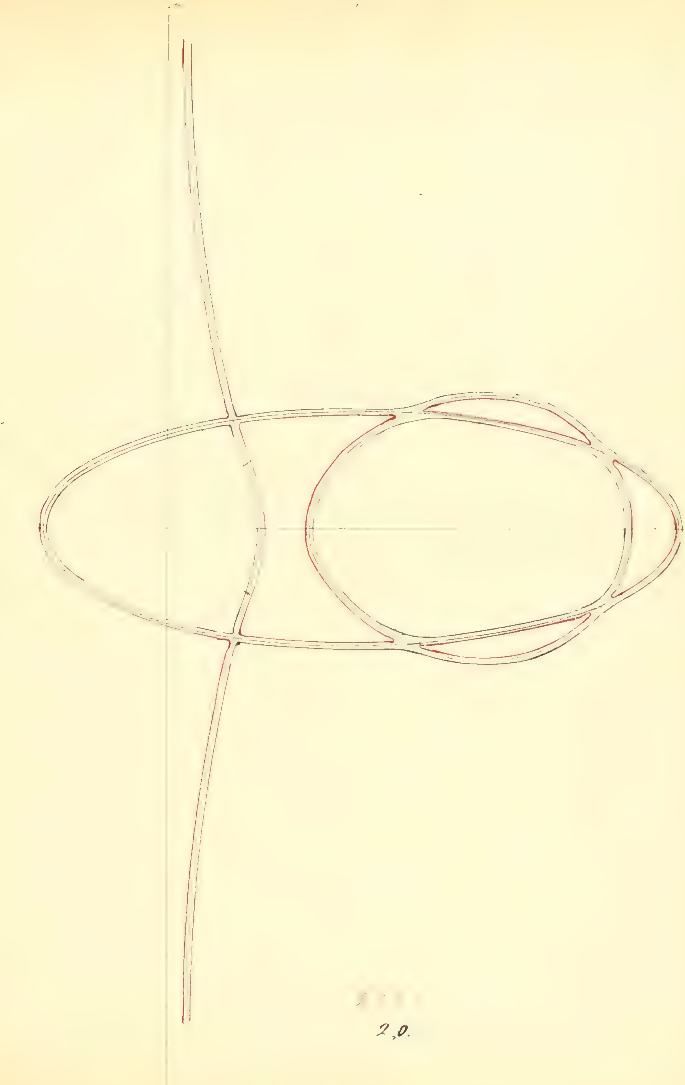
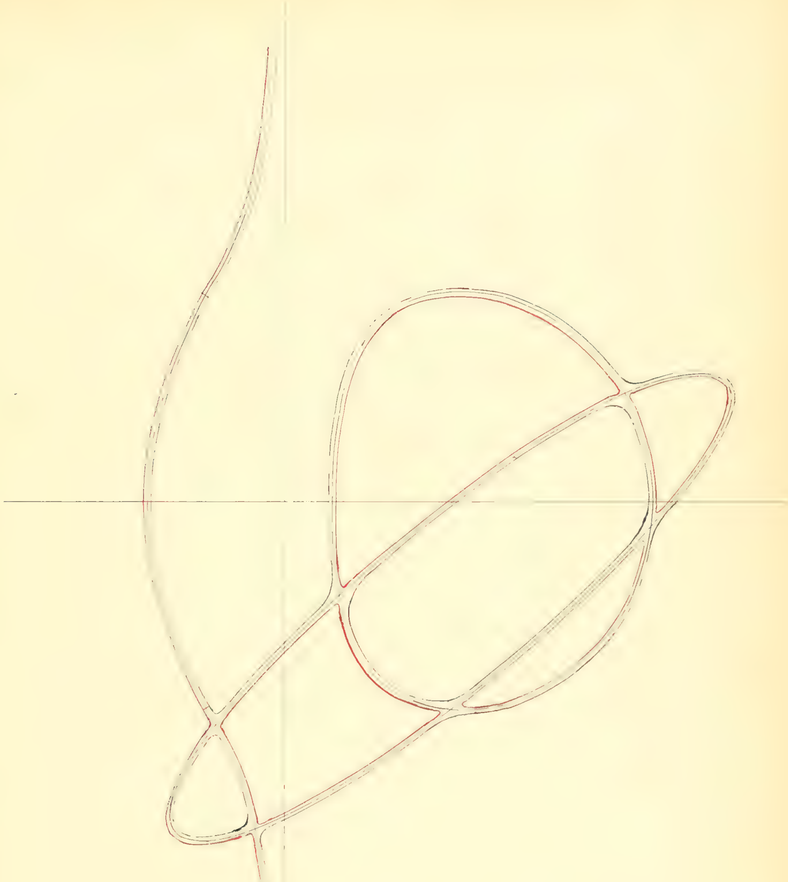


0,0.



2,0.



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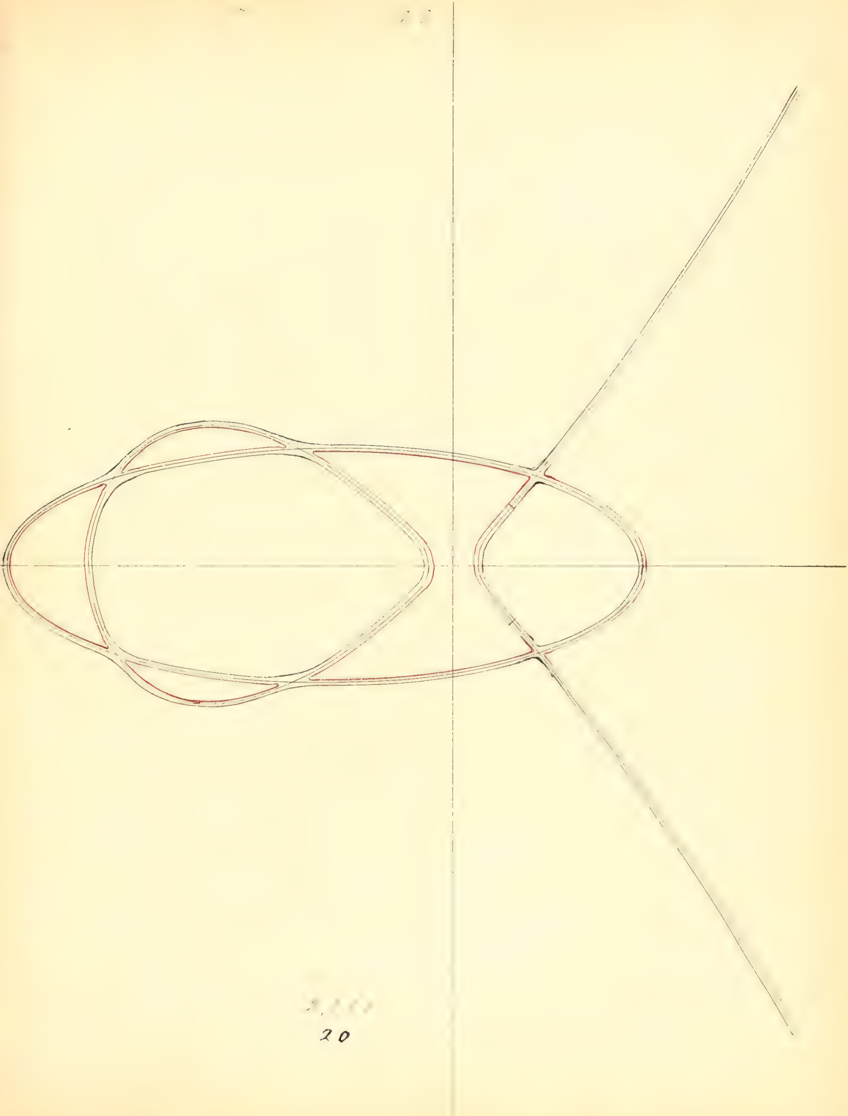

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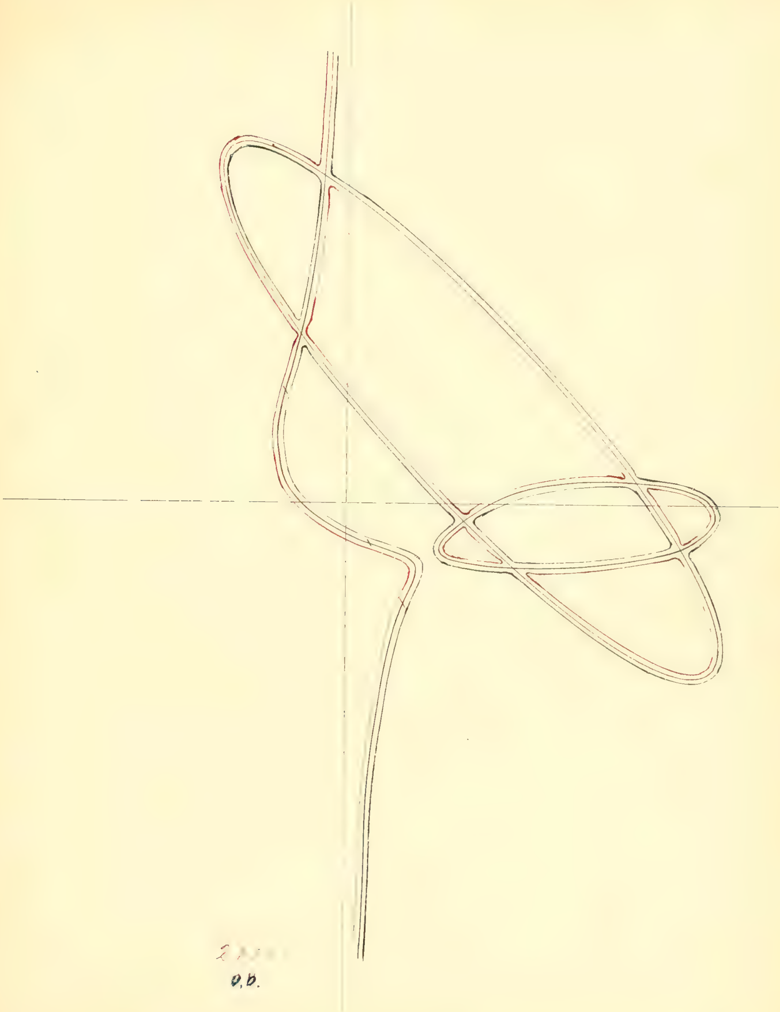
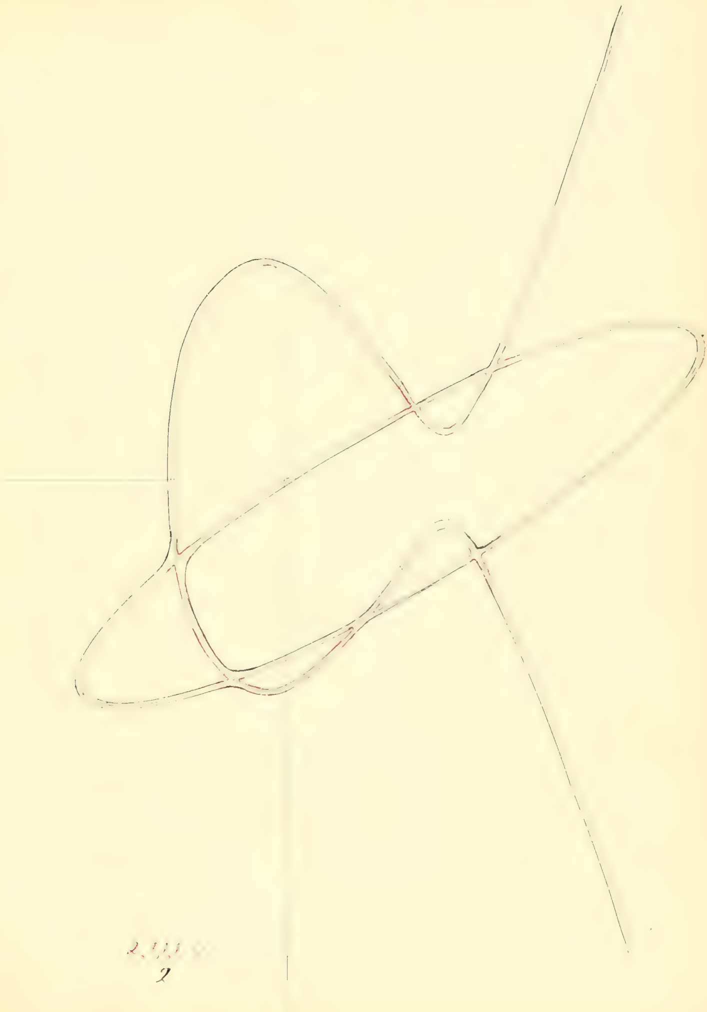


0,b.



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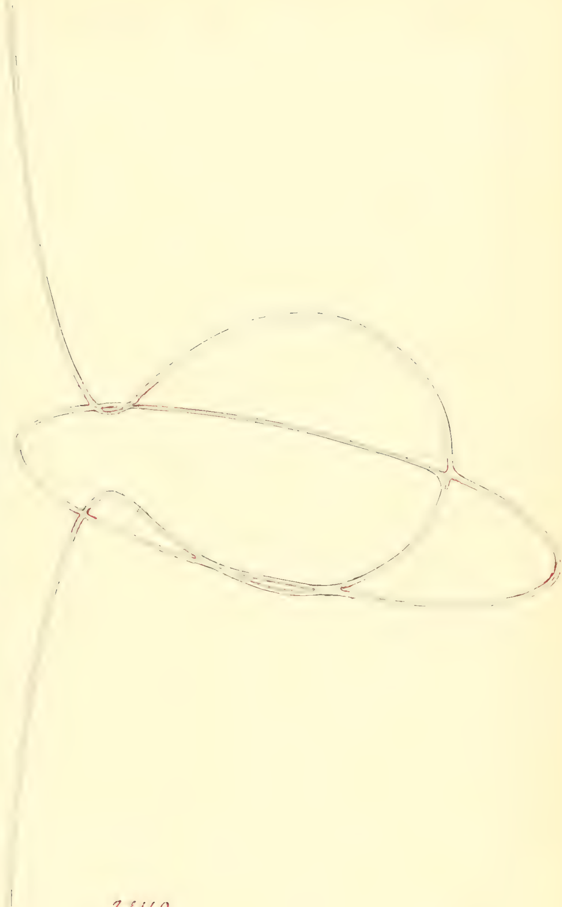
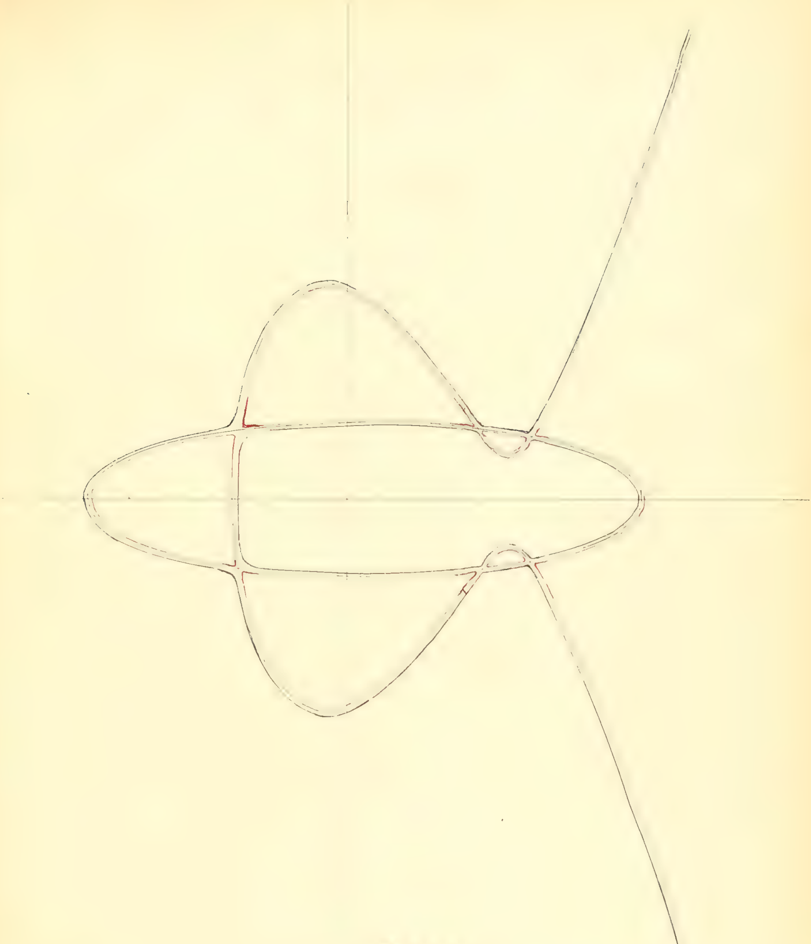


2,1,1,1,0.

2



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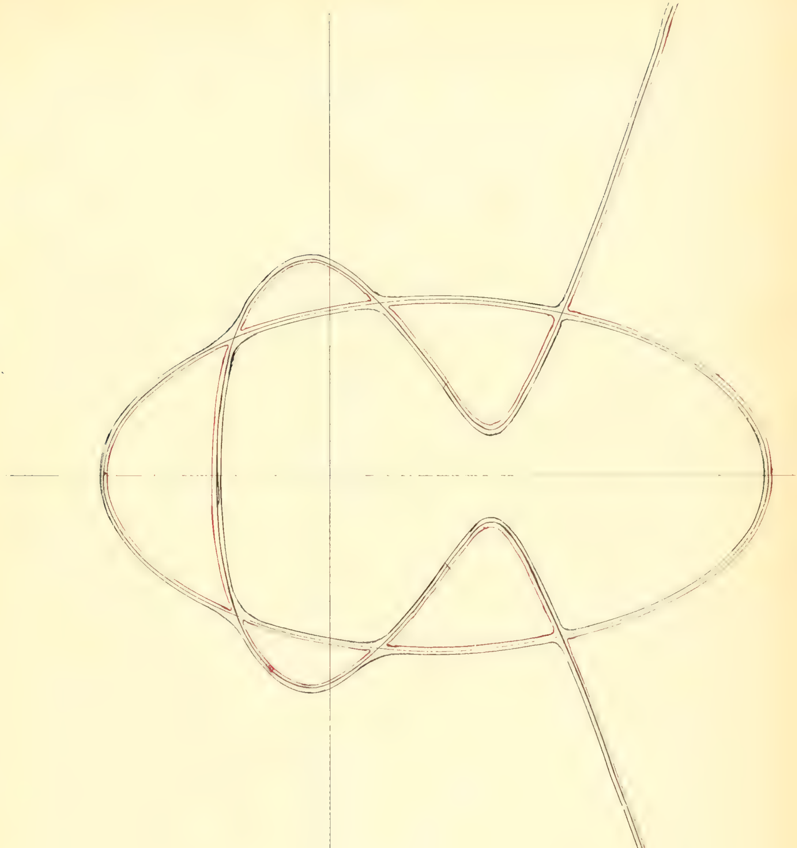

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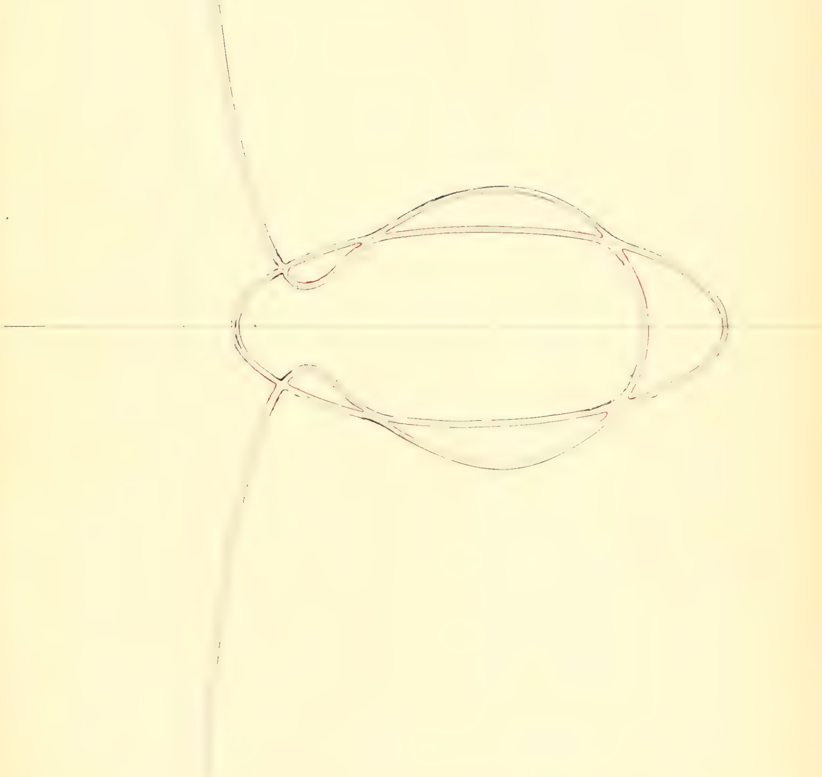

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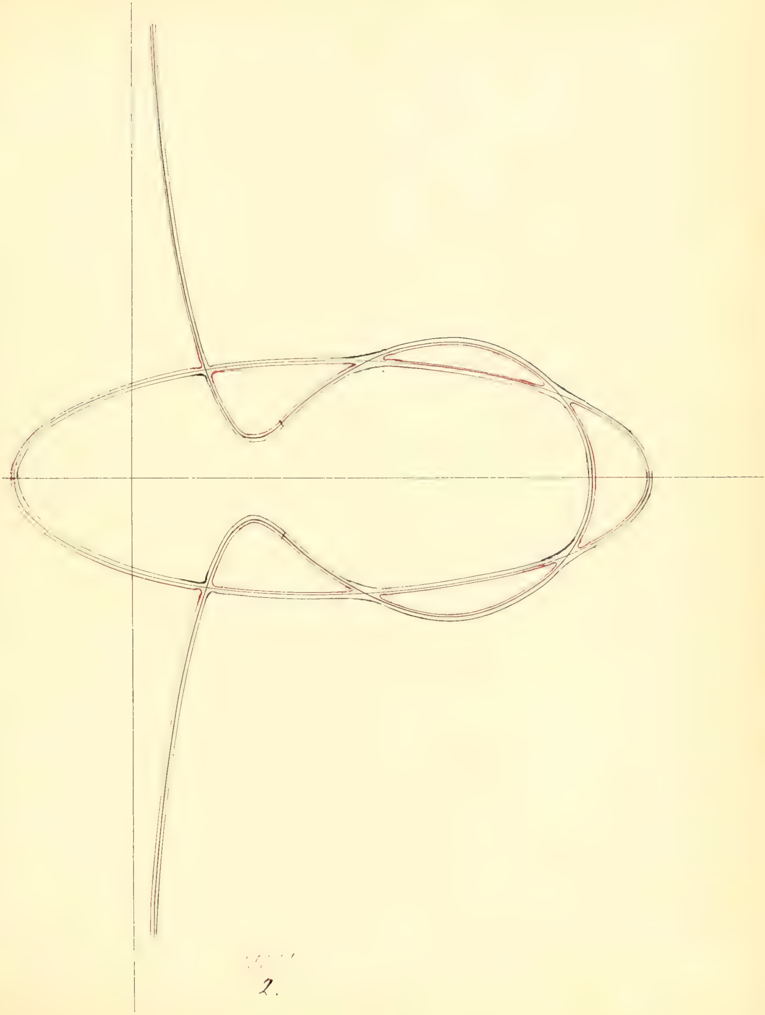

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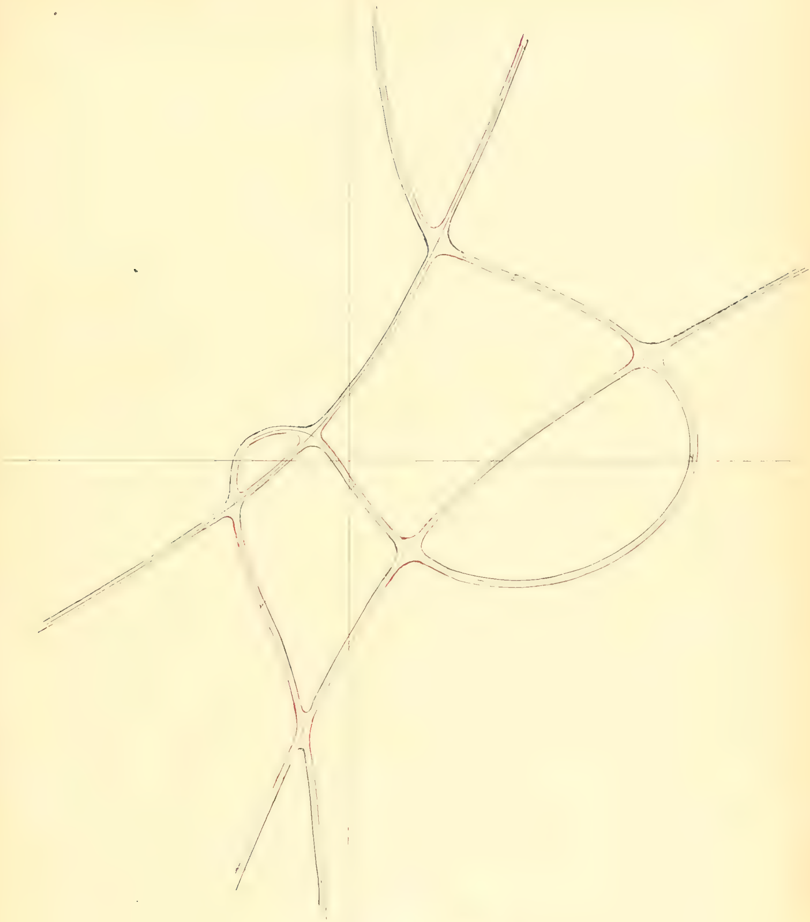

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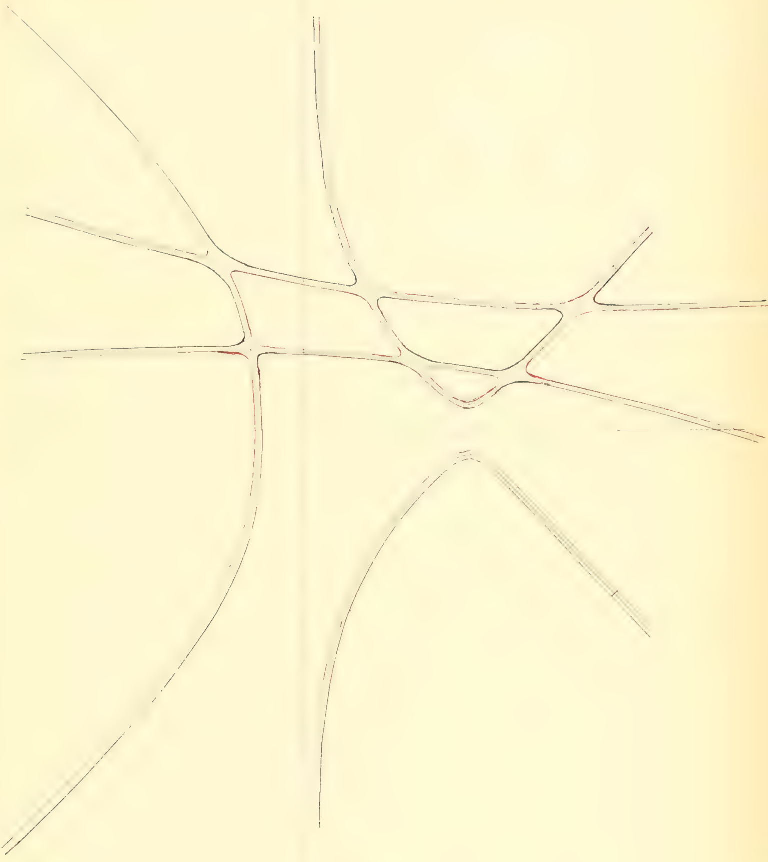

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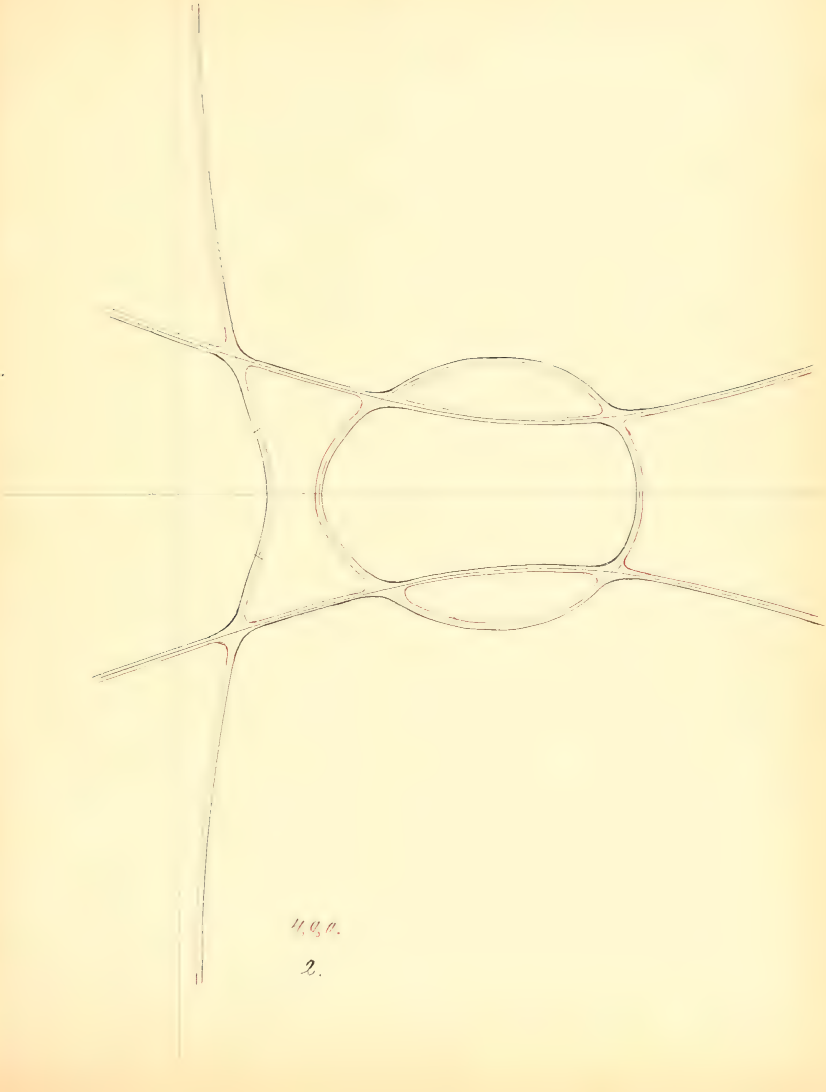

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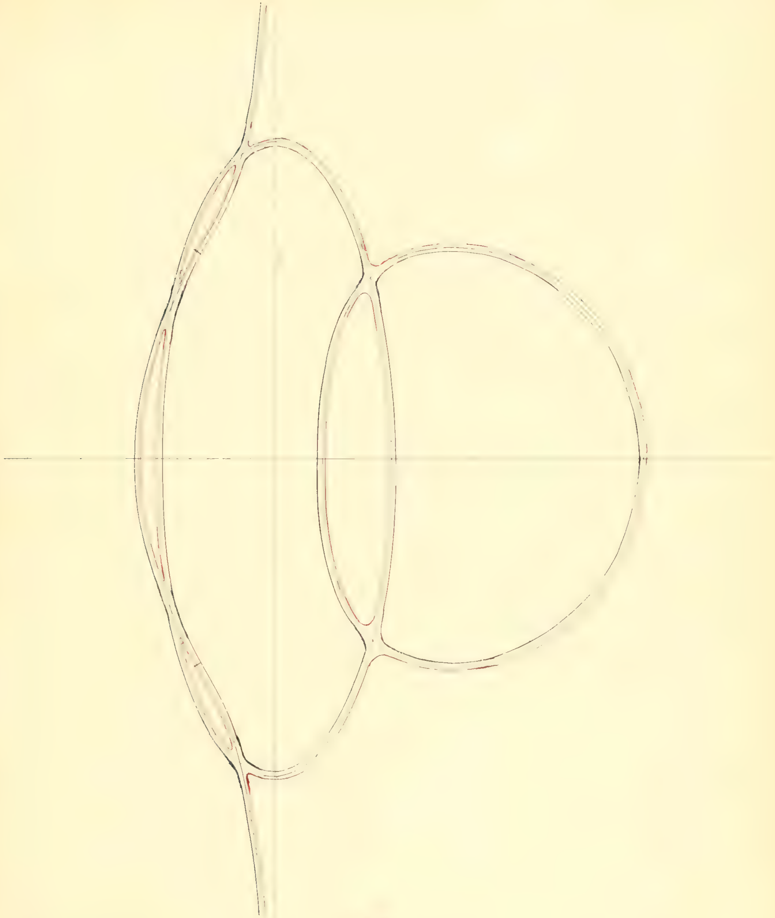

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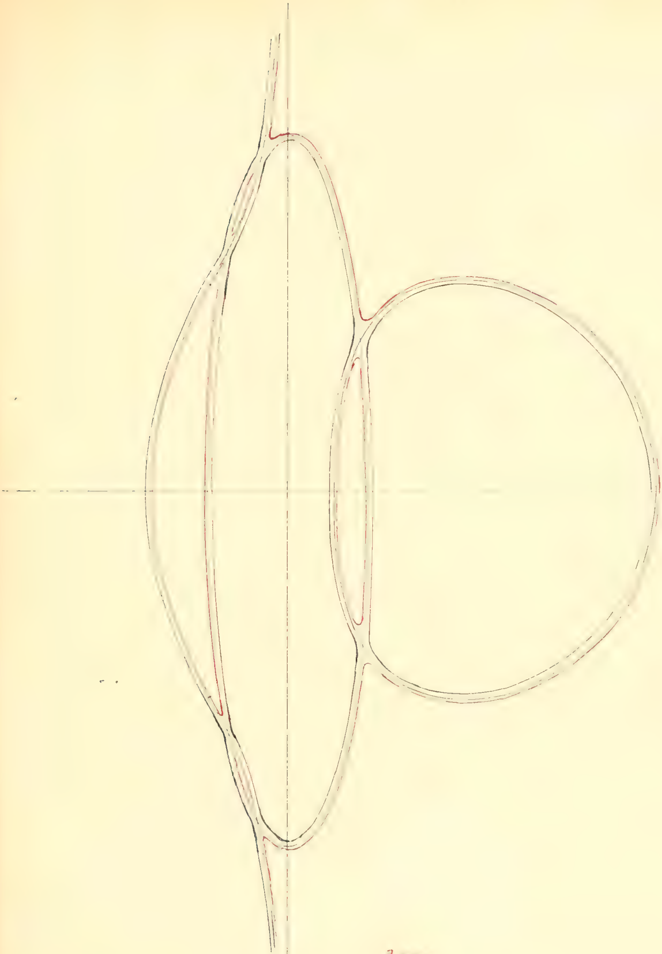

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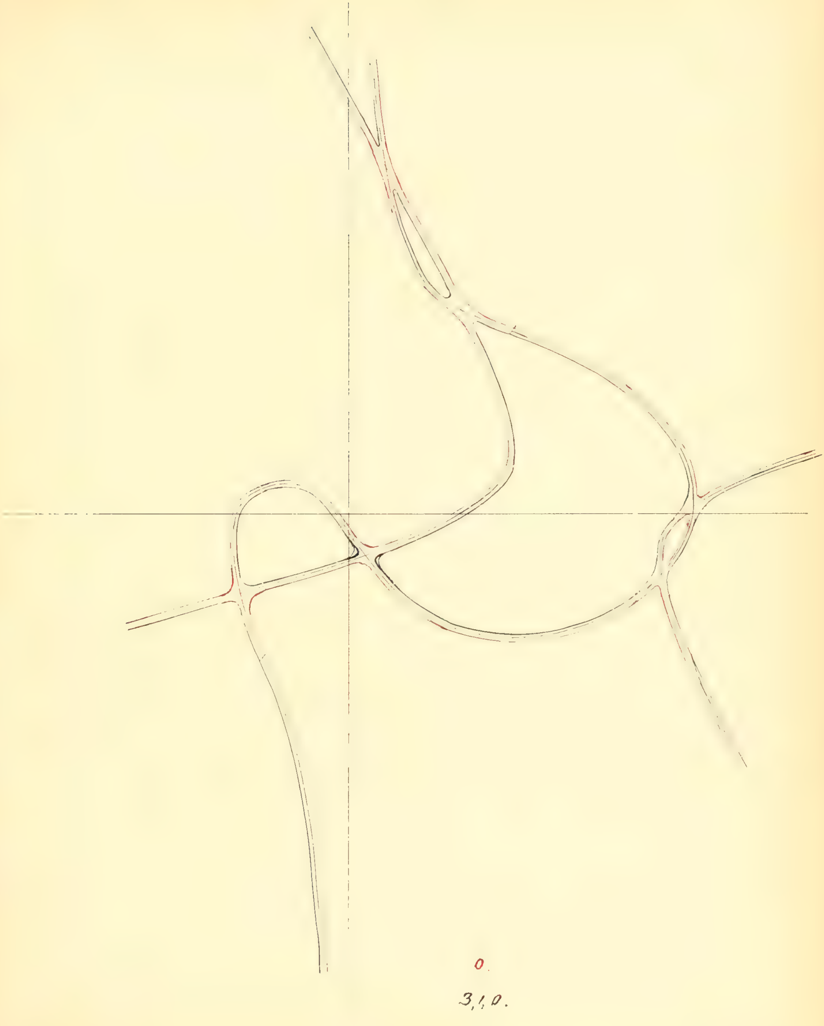

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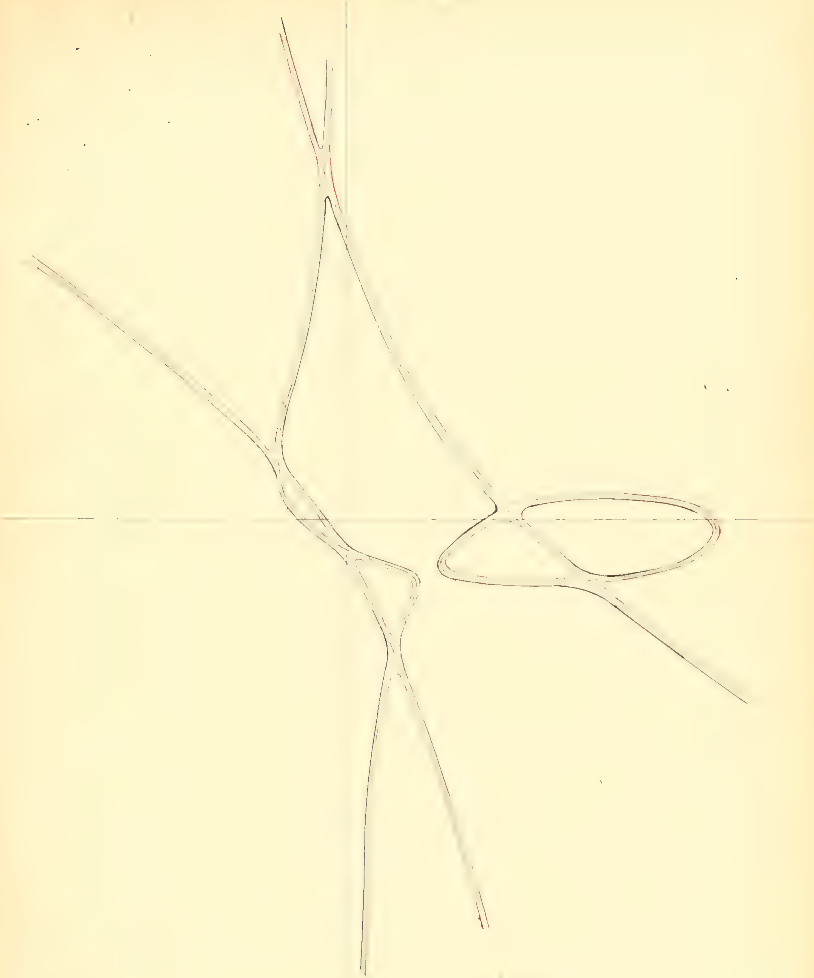

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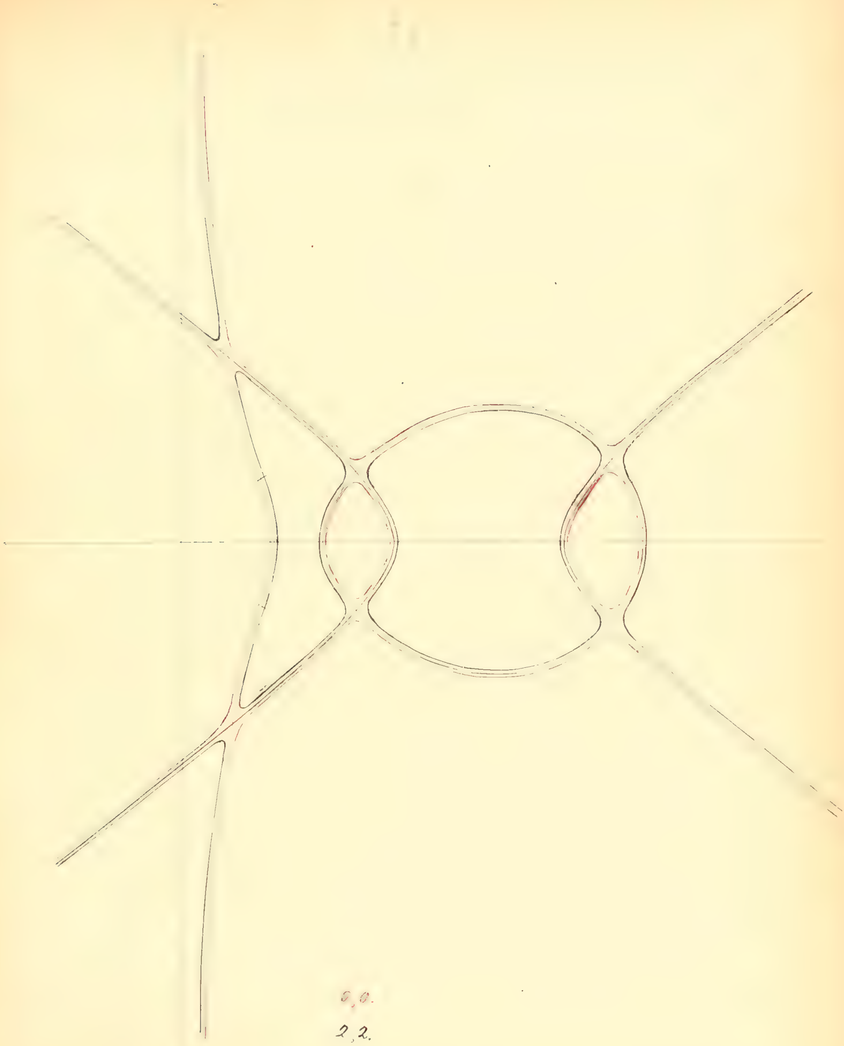

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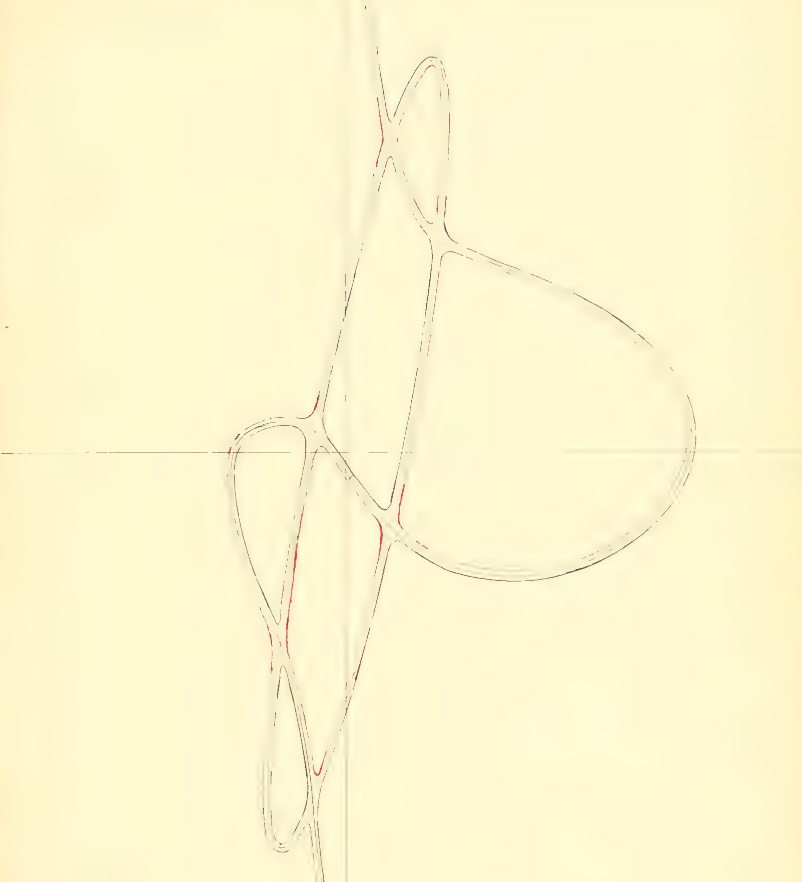

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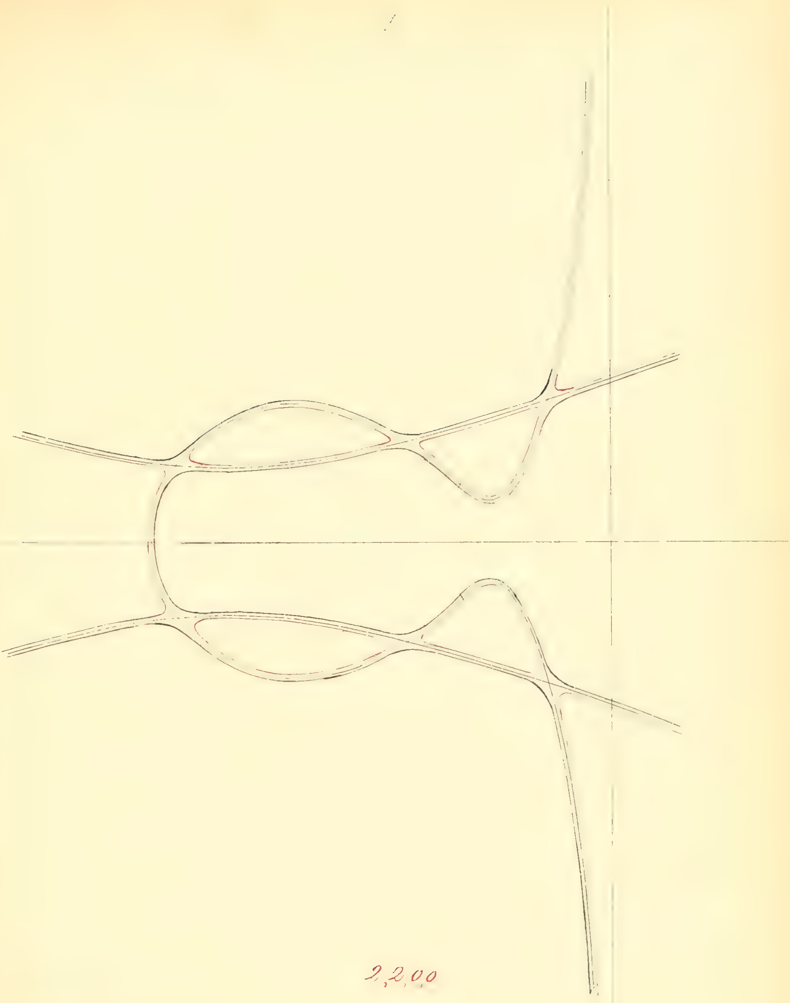

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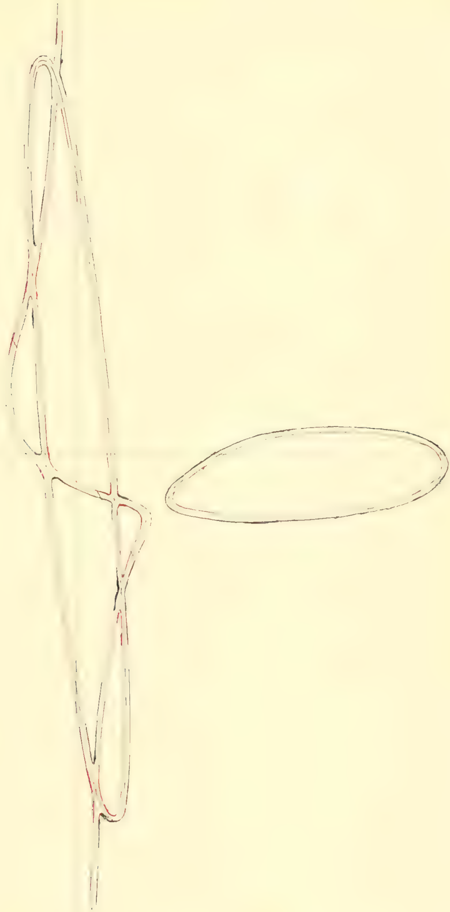

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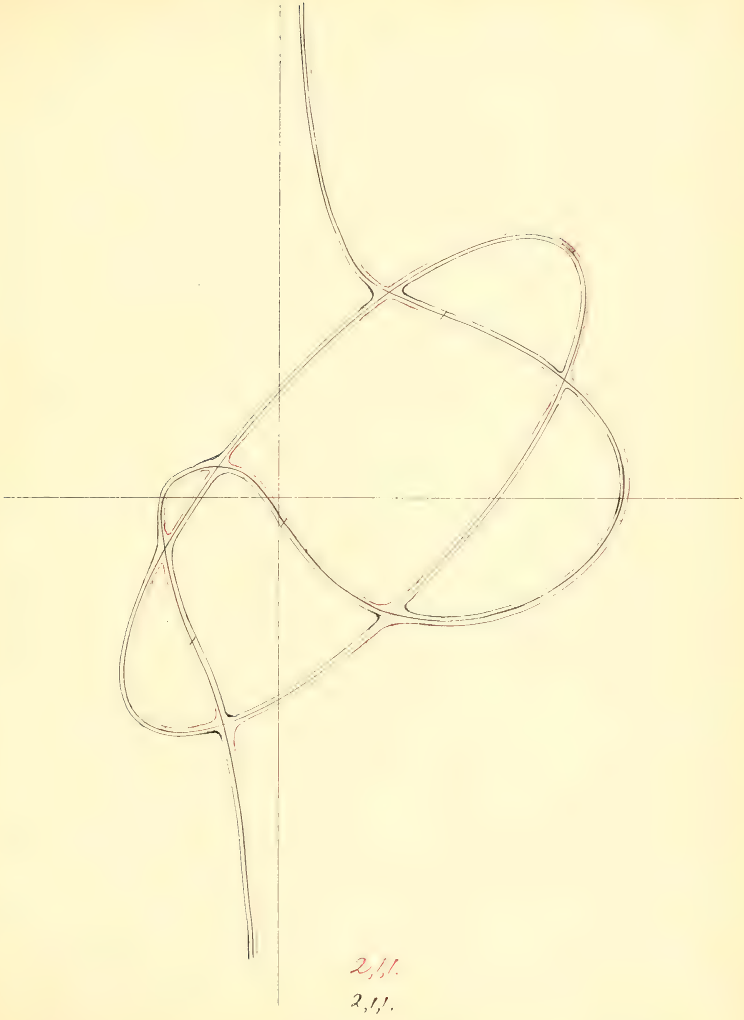

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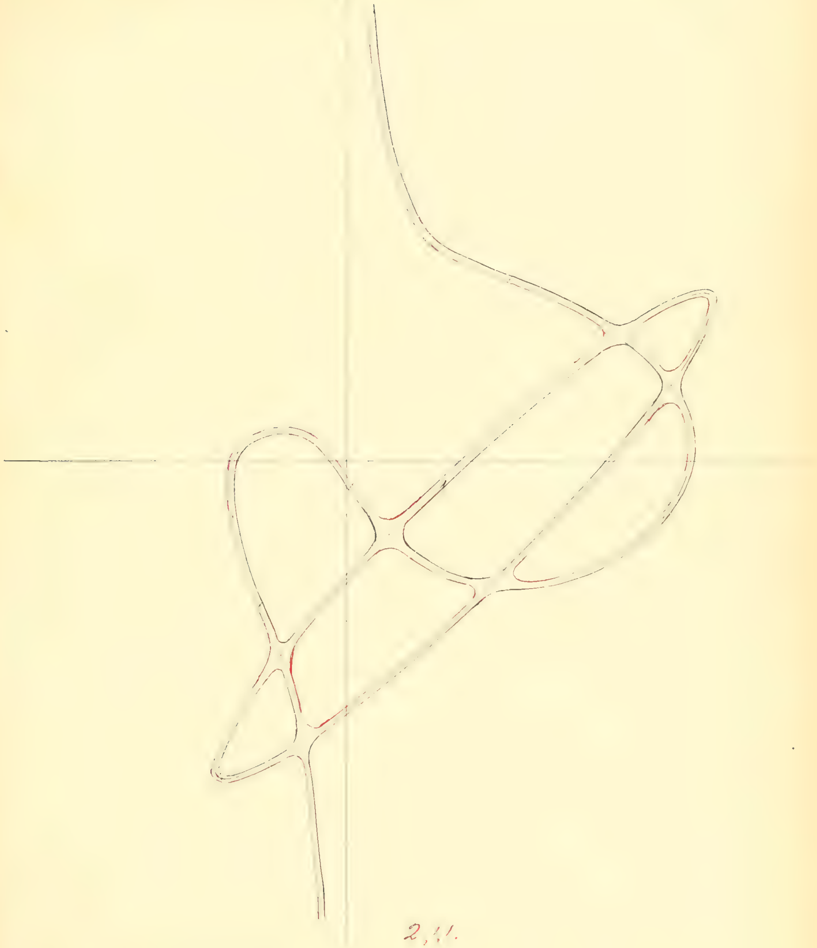

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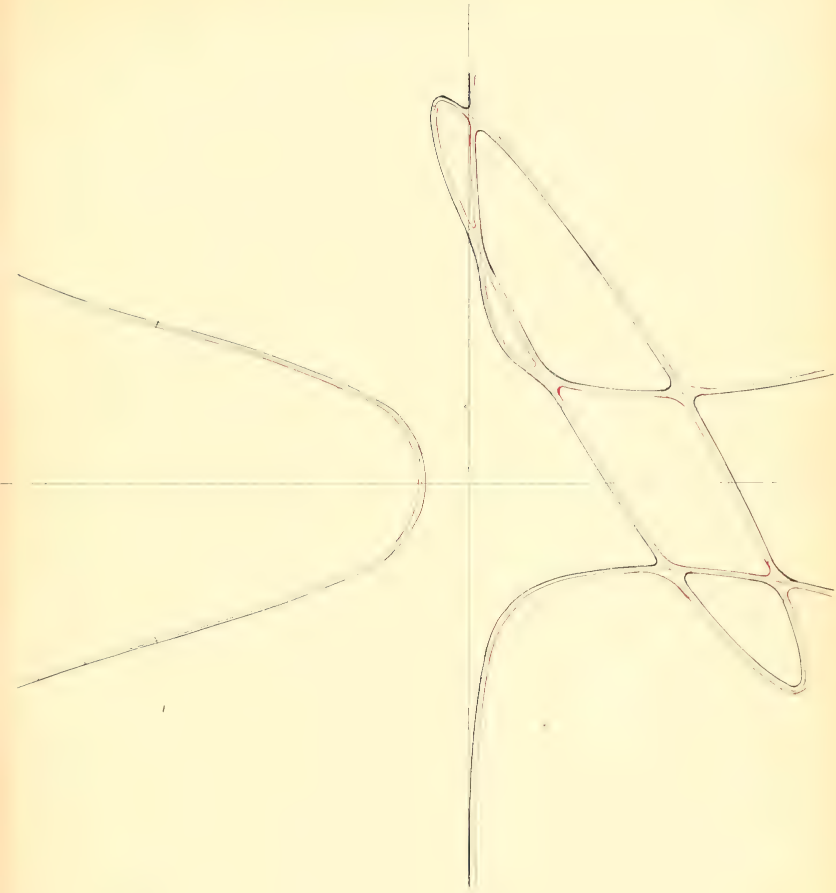

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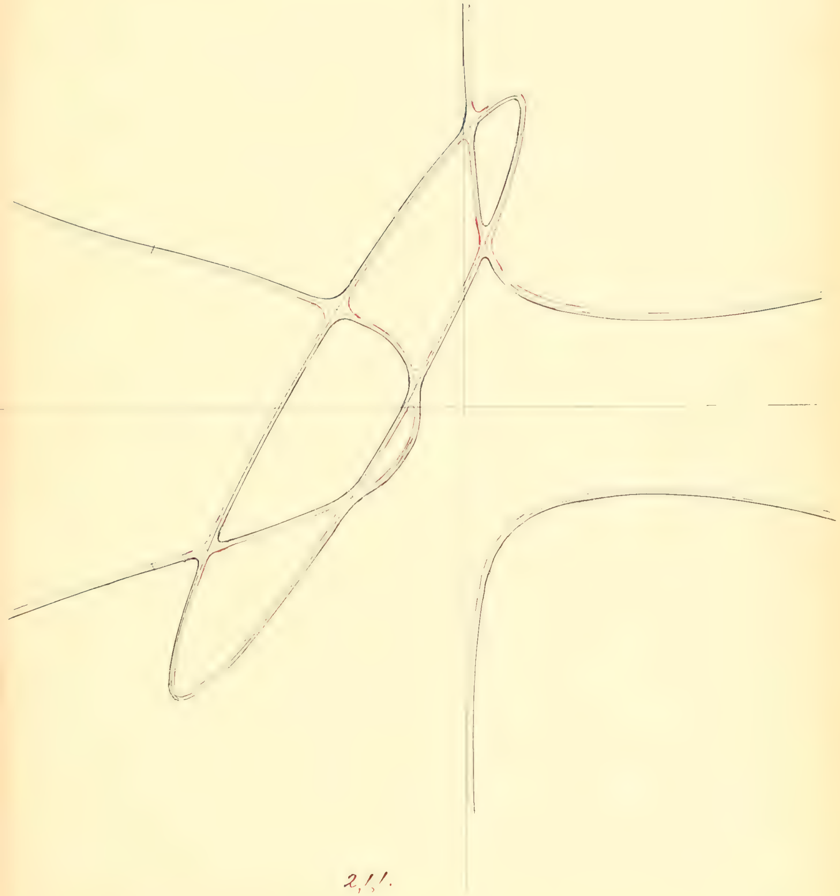

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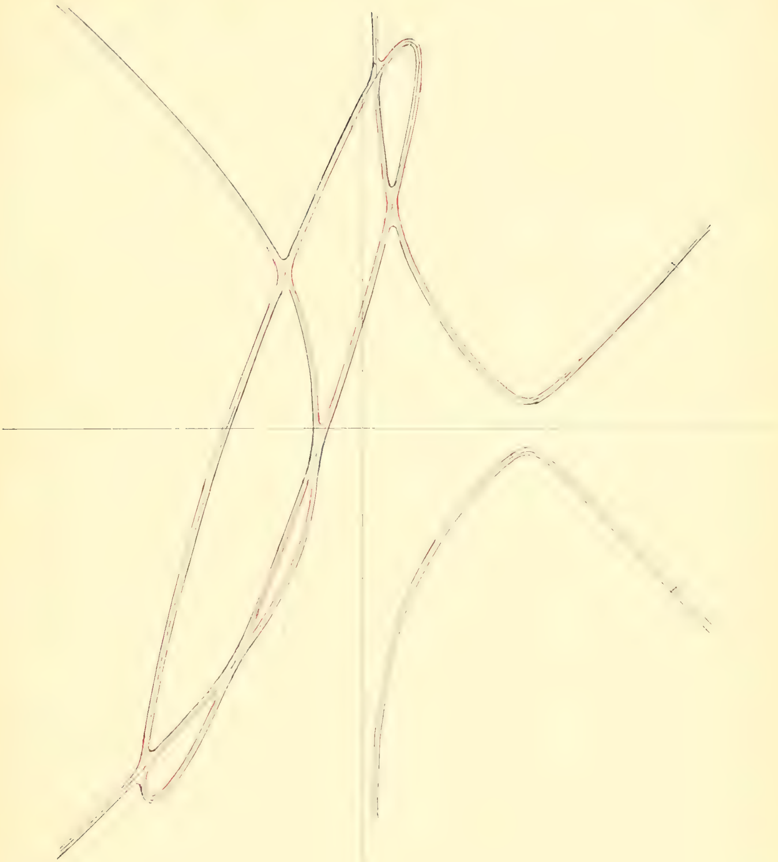

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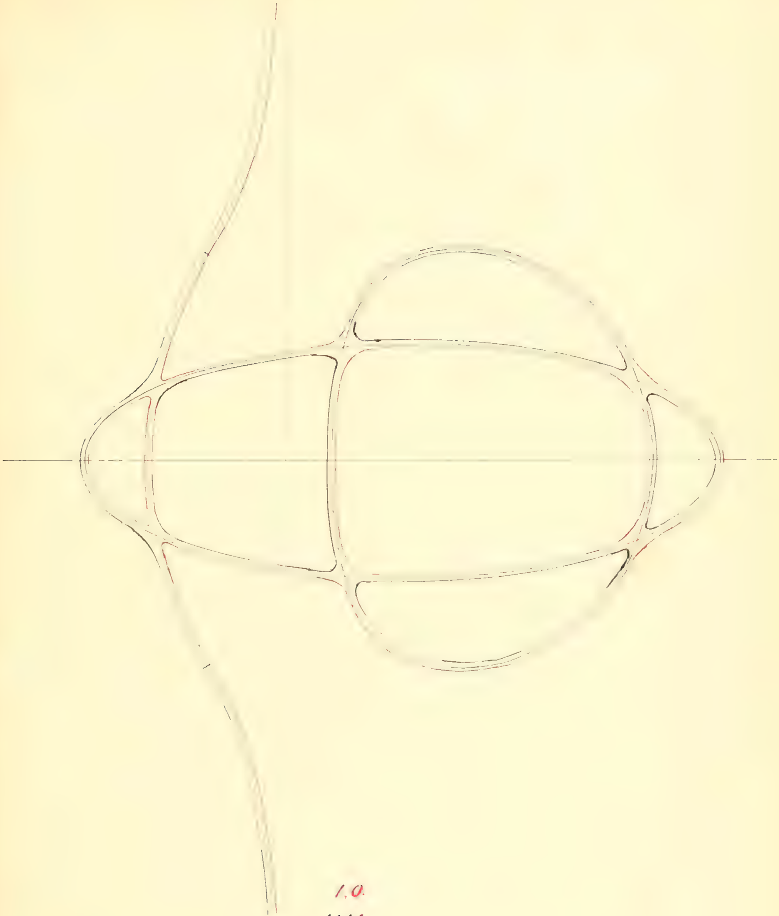

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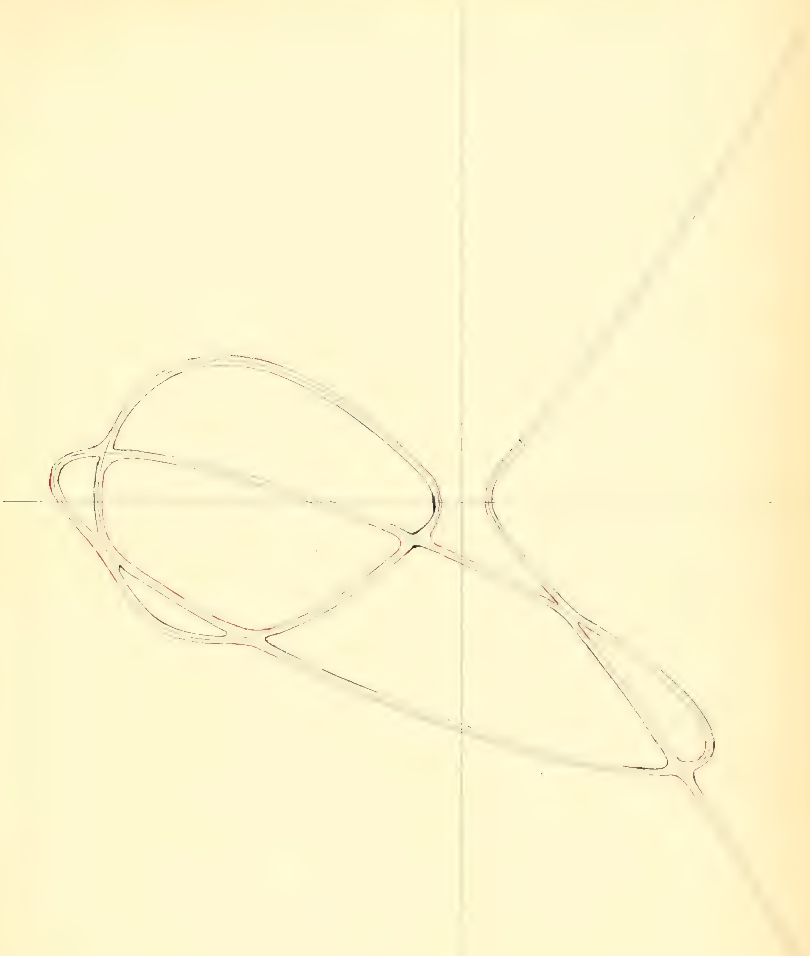

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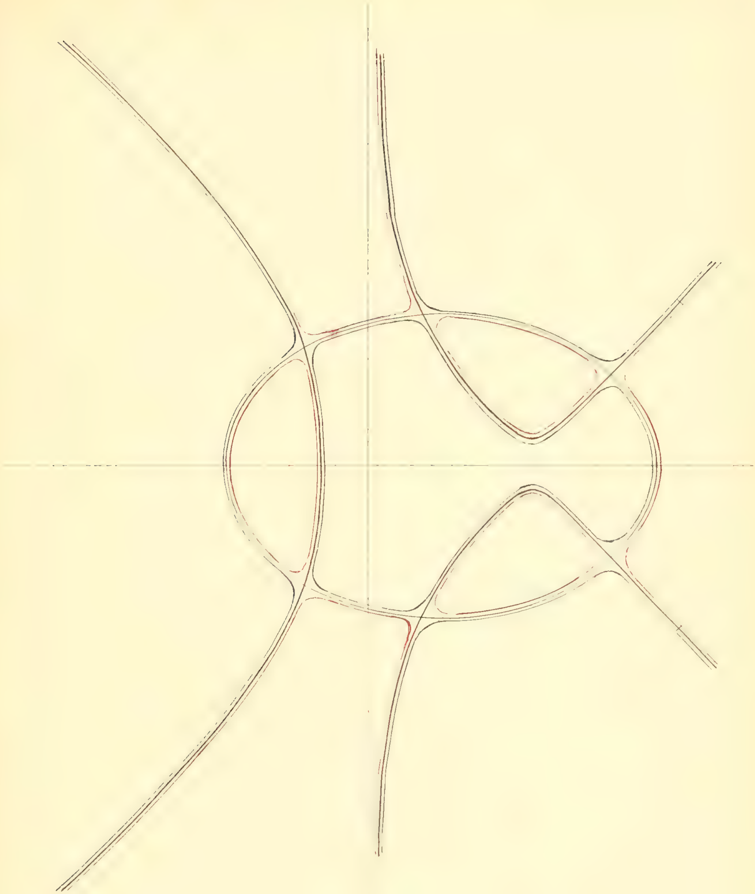

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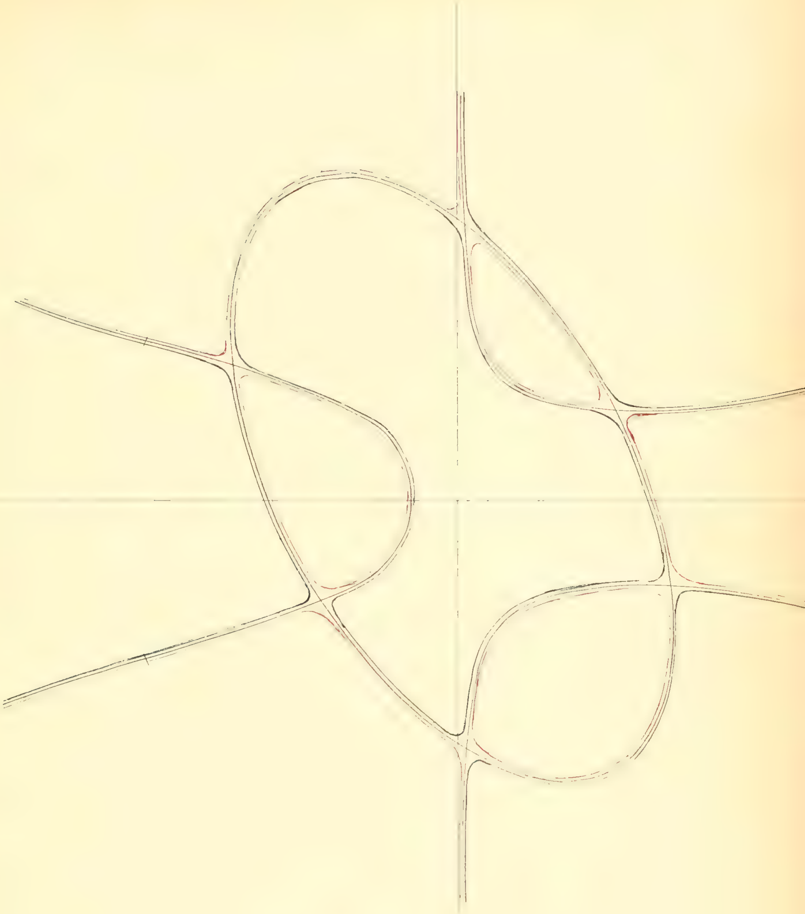

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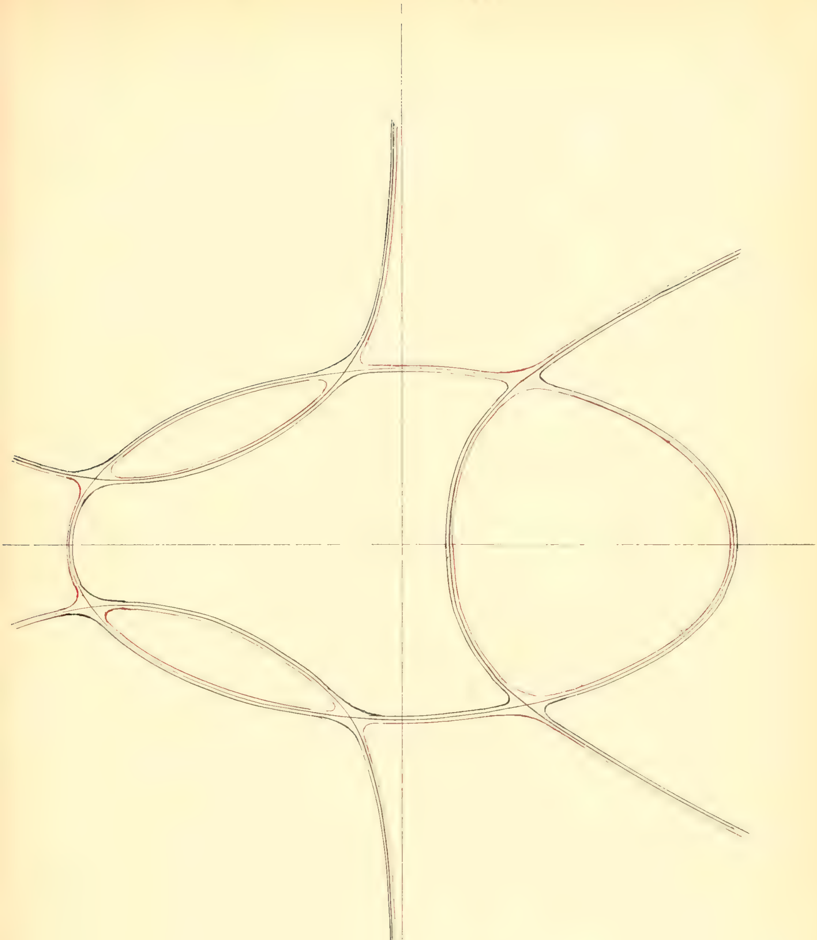

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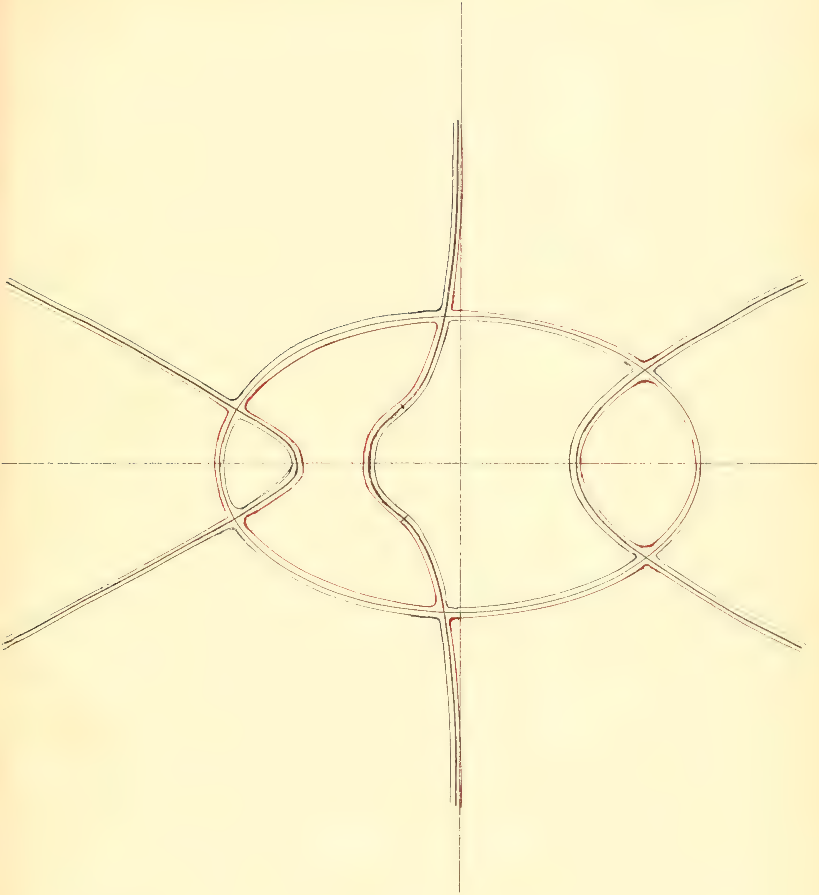

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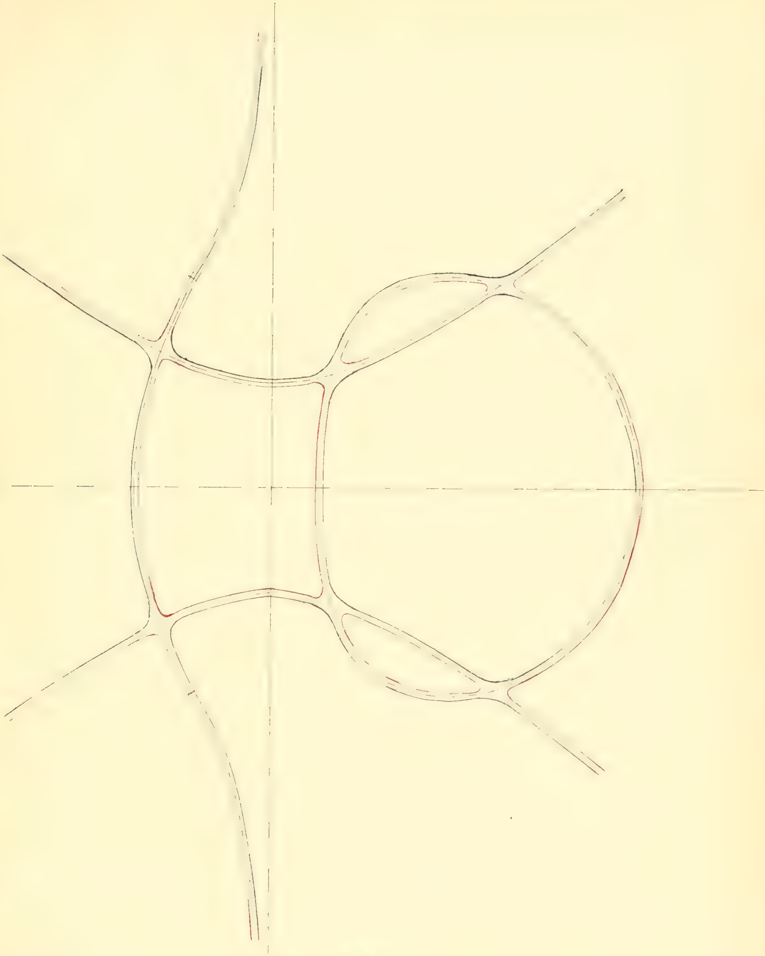

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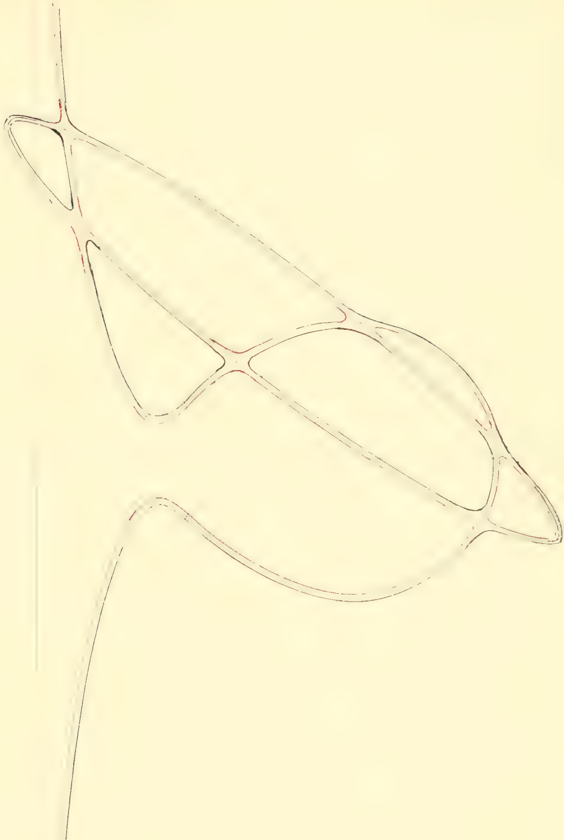

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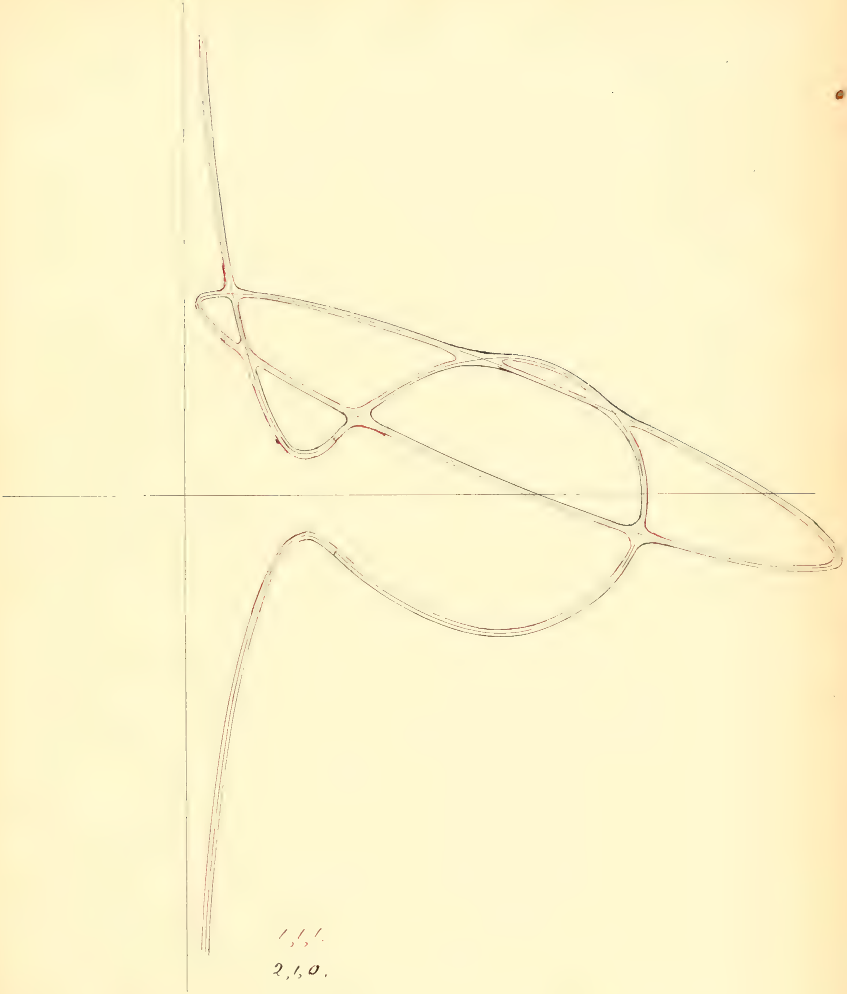
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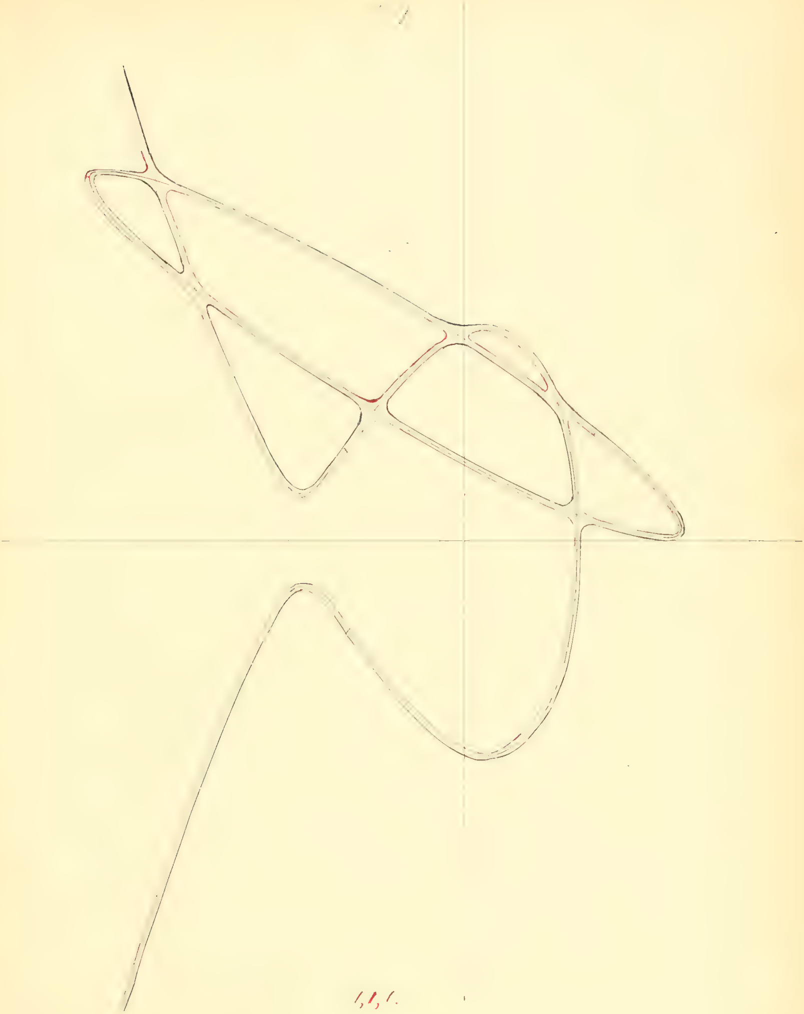

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